# Random Waypoint Model in $n$-Dimensional Space

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**Abstract**

The random waypoint model (RWP) is one of the most widely used mobility models in performance analysis of mobile wireless networks. In this paper we extend the previous work by deriving an analytical formula for the stationary distribution of a node moving according to a RWP model in $n$-dimensional space.

*Key words:* mobility modeling, random waypoint model, ad hoc networking

## 1 Introduction

Random waypoint model (RWP) is one of the most widely used mobility models in performance analysis of wireless ad hoc networks. In the traditional RWP model in $\mathbb{R}^2$, the path of the node is defined by a sequence of random waypoints, $P_1, P_2, \ldots$, placed randomly using a uniform distribution in some convex domain $\mathcal{D} \subset \mathbb{R}^2$. At time $t = 0$ the node is placed at some point $P_0 \in \mathcal{D}$, either randomly using, e.g., uniform distribution or at some fixed starting point. Then the node moves at constant speed $v$ along a line towards the next waypoint $P_1$. Once the node reaches waypoint $P_1$ it takes a new heading towards the next waypoint $P_2$ etc. Each line segment between two waypoints is referred to as a leg and its length is denoted by $|\ell|$.

The RWP model was originally proposed in [2] and has since then been studied extensively. The stationary node distribution in RWP model (in plane) has been studied, e.g., in [3–6]. In [4] Bettstetter et al. present a method to derive an approximate formula for the stationary node distribution in two-dimensional unit square. Using this analysis as a starting point we derived in [1] an exact formula for the node distribution in $\mathbb{R}^2$. In [1], we also analysed the extension of the model where the nodes pause for a random time at the waypoints. In this paper we extend our earlier work in [1] by considering the RWP model in an $n$-dimensional convex set. Using a technique similar to that in [1] we derive an analytical formula for the stationary node distribution in this case. The

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three-dimensional RWP process serves as an elementary model for, e.g. mobile users in an office building or a shopping center. Additionally, it may be a useful mobility model for either airborne or underwater objects.

## 2 Spatial Node Distribution in $\mathbb{R}^3$

For clarity, we first consider the natural extension of RWP model to the three-dimensional space and then generalize the results to an $n$-dimensional space.

Let $V$ denote the volume of the convex domain $\mathcal{D} \subset \mathbb{R}^3$, $\ell$ an arbitrary leg and $\overline{\ell}$ the mean length of a leg. Furthermore, let $f(\mathbf{r})$ denote the pdf of the node location at $\mathbf{r}$. Similarly as in [1,4], we start by considering the length of the intersection of an arbitrary leg $\ell$ and a differential volume element $dV$ at $\mathbf{r}$, denoted by $|\ell \cap dV|$. Note that the mean length of the intersection corresponds to the fraction of time the node spends in $dV$ during a single leg. In particular, referring to Fig. 1 we infer,

$$f(\mathbf{r}) = \frac{1}{\overline{\ell}} \cdot \frac{\mathrm{E}\left[|\ell \cap dV|\right]}{dV}, \qquad (1)$$

$$\mathrm{E}\left[|\ell \cap dV|\right] = \frac{1}{V} \int_{\mathcal{D}} \mathrm{E}\left[|\ell \cap dV||\mathbf{r}_1\right]\ d^3\mathbf{r}_1. \qquad (2)$$

Let $a_1 = a_1(\mathbf{r}, \Omega)$ denote the distance from $\mathbf{r}$ to the boundary of the domain in a given direction $\Omega$ and $a_2$ the distance from $\mathbf{r}$ to the boundary in the opposite direction. From Fig. 1 we deduce that $dV = \Delta \cdot da$ and $da/dA = r^2/(r+a_1)^2$.

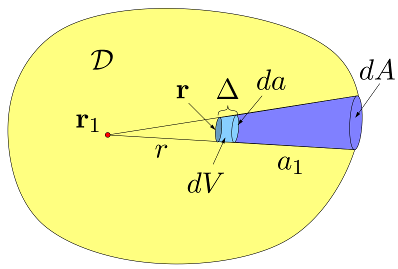


Fig. 1. The integral in Eq. (3) is equal to Δ times the volume of the shaded domain [1,4].

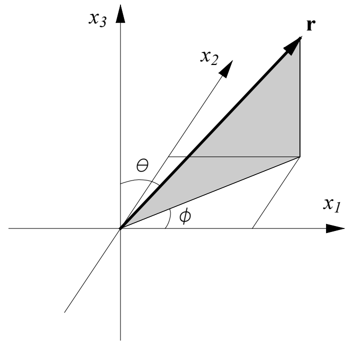


Fig. 2. Spherical Coordinates.

Consequently, [2]

$$\begin{aligned}\mathrm{E}\left[|\ell \cap dV||\mathbf{r}_1\right] &= \frac{1}{V} \int_{\mathcal{D}} |\ell(\mathbf{r}_1, \mathbf{r}_2) \cap dV|\ d^3\mathbf{r}_2 \\ &= \frac{1}{3V} \Delta \cdot \left((r+a_1) \cdot dA - r \cdot da\right) \\ &= \frac{\Delta}{3V} \cdot \left(\frac{(r+a_1)^3 - r^3}{r^2}\right)\ da \\ &= \frac{dV}{3V} \cdot \frac{(r+a_1)^3 - r^3}{r^2}. \qquad (3)\end{aligned}$$

Substituting Eqs. (2) and (3) back into Eq. (1) yields,

$$f(\mathbf{r}) = \frac{1}{3\overline{\ell}V^2} \int_{\mathcal{D}} \frac{(r+a_1)^3 - r^3}{r^2}\ d^3\mathbf{r}_1$$

[2] The volume of 3-dimensional cone is $1/3 \times$ height $\times$ area of the base, and the shaded domain corresponds to the difference between two cones.

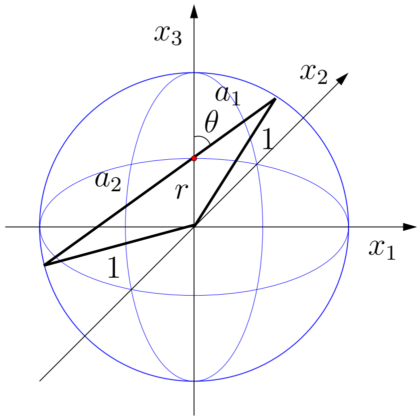


Fig. 3. Due to symmetry in a unit sphere one can consider points $(0,0,r)$ for which $a_1$ and $a_2$ are a function of $r$ and $\theta$.

$$= \frac{1}{3\bar{\ell}V^2}\int_\Omega d\Omega \int_0^{a_2} dr \, \left((r+a_1)^3 - r^3\right)$$
$$= \frac{1}{12\bar{\ell}V^2}\int_\Omega d\Omega \, \left((a_1+a_2)^4 - (a_1^4+a_2^4)\right).$$

More specifically, using spherical coordinates we get (see Fig. 2)

$$f(\mathbf{r}) = \frac{1}{6\bar{\ell}V^2}\int_0^\pi d\theta \, \sin\theta \int_0^\pi d\phi \, H(\mathbf{r},\theta,\phi). \quad (4)$$

where

$$H(\mathbf{r},\theta,\phi) = (a_1+a_2)^4 - (a_1^4+a_2^4),$$

and the distances $a_1$ and $a_2$ depend on the shape of the particular domain $\mathcal{D}$.

## 3 Spatial Node Distribution in $\mathbb{R}^n$

The same procedure can be generalized to $n$ dimensions. The "volume" of an $n$-dimensional cone is

$$V_n(h) = \frac{h \cdot A}{n},$$

where $h$ is the height and $A$ corresponds to the "area" of the base. Let $\mathcal{D} \subset \mathbb{R}^n$ be a convex set with volume $V$. Then it is easy to see that the stationary distribution of the RWP process in $\mathcal{D}$ is given by

$$f(\mathbf{r}) = \frac{1}{n\bar{\ell}V^2}\int_{\mathcal{D}} \frac{(r+a_1)^n - r^n}{r^{n-1}} \, d^n\mathbf{r},$$

which can be expressed as

$$f(\mathbf{r}) = \frac{1}{n(n+1)V^2\bar{\ell}}\int_\Omega d\Omega \, H(\mathbf{r},\Omega). \quad (5)$$

where

$$H(\mathbf{r},\Omega) = (a_1+a_2)^{n+1} - (a_1^{n+1}+a_2^{n+1}).$$

The mean length of leg $\bar{\ell}$ is obtained from the normalization condition $\int f(\mathbf{r}) \, d^n\mathbf{r} = 1$,

$$\bar{\ell} = \frac{1}{n(n+1)V^2}\int_{\mathcal{D}} d^n\mathbf{r} \int_\Omega d\Omega \, H(\mathbf{r},\Omega). \quad (6)$$

## 4 Examples

*Example 1: RWP model in plane*: For $n = 2$ the general expression (5) yields

$$f(\mathbf{r}) = \frac{1}{6\bar{\ell}V^2}\int_0^{2\pi} (a_1+a_2)^3 - (a_1^3+a_2^3) \, d\theta$$
$$= \frac{1}{\bar{\ell}V^2}\int_0^\pi a_1 a_2 (a_1+a_2) \, d\theta,$$

which is identical to the equation derived in [1].

*Example 2: Unit sphere in* $\mathbb{R}^3$: Due to the symmetry, the pdf is a function of distance $r = |\mathbf{r}|$ only and without loss of generality we can consider the point $\mathbf{r} = (0,0,r)$. From Fig. 3 one immediately obtains that

$$a_1(r,\theta) = \sqrt{1-r^2\sin^2\theta} - r\cdot\cos\theta,$$
$$a_2(r,\theta) = \sqrt{1-r^2\sin^2\theta} + r\cdot\cos\theta. \tag{7}$$

Using the 3-dimensional expression (4) we get

$$f(\mathbf{r}) = \frac{3}{16\pi\cdot\bar{\ell}}\int_0^\pi \sin\theta\cdot a_1a_2(2a_1^2+3a_1a_2+2a_2^2)\,d\theta.$$

Writing

$$\begin{aligned} h(r,\theta) &= \sin\theta\cdot a_1a_2(2a_1^2+3a_1a_2+2a_2^2) \\ &= \sin\theta\cdot(1-r^2)(7-3r^2+4r^2\cos 2\theta) \end{aligned}$$

and

$$h(r) = \int_0^\pi h(r,\theta)\,d\theta = \frac{2}{3}\left(21-34r^2+13r^4\right)$$

we get

$$\int_0^1 4\pi r^2\cdot h(r)\,dr = \frac{192\pi}{35}.$$

Thus the mean length of leg $\bar{\ell}$ and the pdf of the node location at $\mathbf{r}$ are

$$\begin{aligned} \bar{\ell} &= \frac{36}{35} \approx 1.029, \\ f(\mathbf{r}) &= \frac{35}{288\pi}\cdot\left(21-34r^2+13r^4\right). \end{aligned}$$

In the spherically symmetric case, the pdf of the random variable $r = |\mathbf{r}|$, denoted by $f_d(r)$, is $f_d(r) = 4\pi r^2 f(\mathbf{r})$,

$$f_d(r) = \frac{35}{72}\cdot r^2\left(21-34r^2+13r^4\right).$$

The cumulative pdf for the unit sphere is illustrated in Fig. 4, where each section represents a probability mass of 0.2.

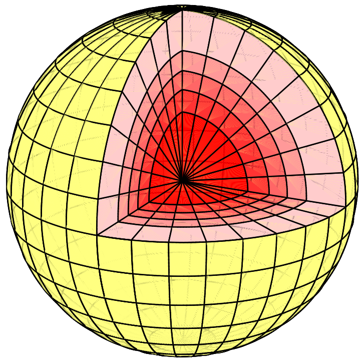

Fig. 4. Stationary node distribution of the RWP process in the unit sphere. Each section represents a probability mass of 0.2.

*Example 3: Unit hypersphere in* $\mathbb{R}^n$: The area and volume of the $n$-dimensional unit hypersphere are [7]

$$A_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \qquad \text{and} \qquad V_n = \frac{\pi^{n/2}}{\Gamma(n/2+1)}.$$

Again, due to the symmetry, the pdf is a function of distance $r$ only and we can arbitrarily take $\mathbf{r} = (0,\ldots,0,r)$. Eq. (7) holds still with $\theta$ being the angle between $\mathbf{r}$ and the $x_n$-axis. Differential surface area of the hypersphere between $\theta$ and $\theta+d\theta$ is $A_{n-1}\sin^{n-2}\theta\,d\theta$. Hence, we have

$$f(\mathbf{r}) = \frac{A_{n-1}}{n(n+1)V_n^2\cdot\bar{\ell}}\cdot h(r),$$

where

$$h(r) = \int_0^\pi \sin^{n-2}\theta\cdot H(r,\theta)\,d\theta$$

with

$$H(r,\theta) = \left((a_1+a_2)^{n+1} - (a_1^{n+1}+a_2^{n+1})\right).$$

In Fig. 5 the pdf of the node location at $\mathbf{r}$, $f(\mathbf{r})$, and the pdf of the distance $r$ from the

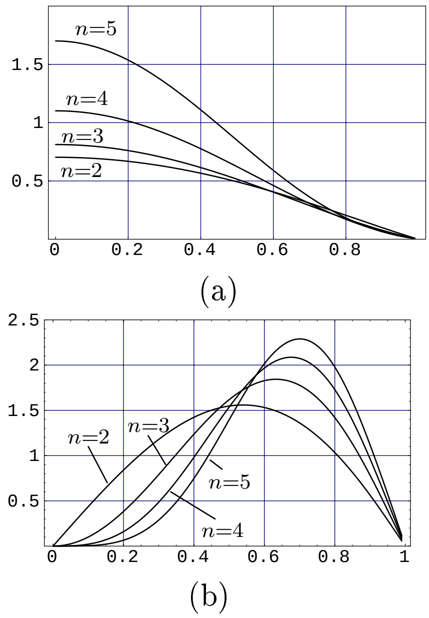


Fig. 5. Pdf of the node location (left) and distance from the origin (right) in an $n$-dimensional unit hypersphere, $n = 2, 3, 4, 5$.

origin, $f_d(r)$, are depicted for dimensions $n = 2, 3, 4, 5$. The maximum value of the pdf $f(r)$ attained at the center of the hypersphere increases as the dimension $n$ increases. Furthermore, from the Fig. 5(b) it can be noted that as the dimensions increase the probability mass of the random variable $r$ shifts towards the surface, $r = 1$. Finally, the mean length of leg given by Eq. (6) can be written for the unit hypersphere as

$$\bar{\ell}_n = \frac{A_{n-1}A_n}{n(n+1)V_n^2} \int_0^1 r^{n-1} h(r)\, dr.$$

For $n = 1, \ldots, 5$ we obtain the well-known results [8,9]:

$$\begin{aligned} &\bar{\ell}_1 = \tfrac{2}{3} \approx 0.667, \quad &&\bar{\ell}_4 = \tfrac{16384}{4725\pi} \approx 1.104, \\ &\bar{\ell}_2 = \tfrac{128}{45\pi} \approx 0.905, \quad &&\bar{\ell}_5 = \tfrac{800}{693} \approx 1.154. \\ &\bar{\ell}_3 = \tfrac{36}{35} \approx 1.029, && \end{aligned}$$

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