Modeling Dynamics of Exponentially Averaged Queue

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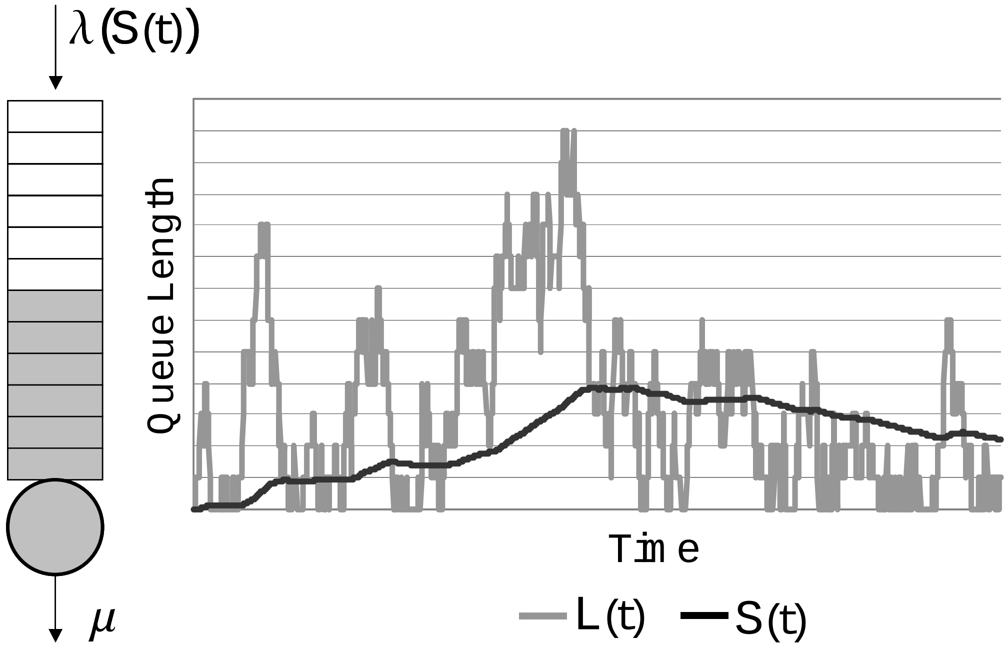
Introduction

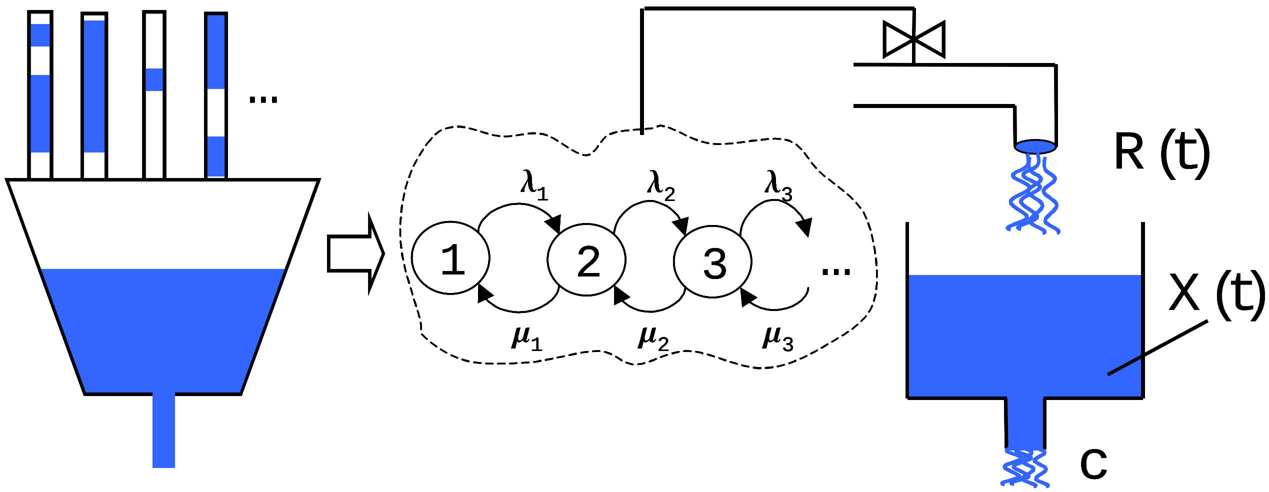


- Instantaneous queue length $L(t), L(t) \in \{0, 1, ..., K\}$
- Exponentially averaged queue length $S(t)$, $\frac{d}{dt}S(t) = -\alpha(S(t) - L(t))$
- Poisson arrivals, packet drop probability depends on $S(t)$
- Exponential service times with parameter μ

Introduction (continued)

- Motivation for modeling dynamics of exponentially averaged queue
  - RED mechanism
  - DiffServ architecture: Assured Forwarding PHB
- The model provides information about
  - The stationary properties of exponentially averaged queue length (PDF)
  - The average packet drop probability
- Results may be utilized to
  - Determine the parameters of RED/AF buffers

Background: Fluid Queues

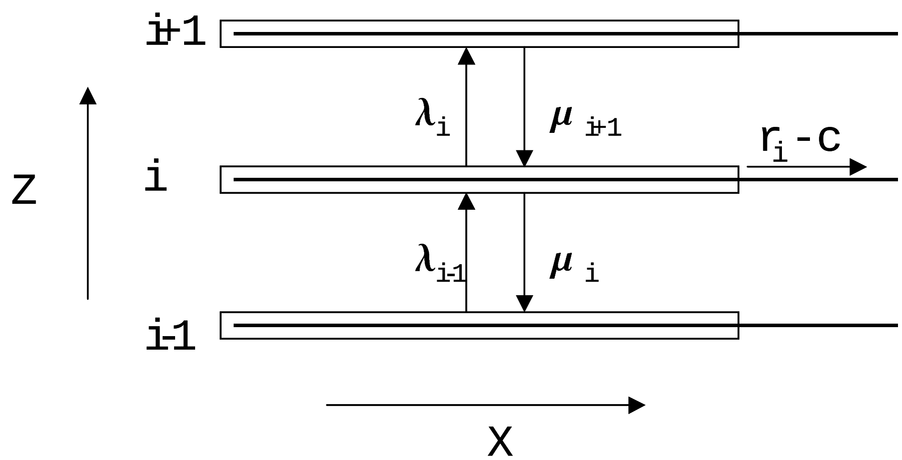


- Arriving work is regarded as continuous fluid flow
  - Arrival intensity $R(t)$
  - Drain rate c
  - The amount of fluid $X(t)$ in the container changes with rate $R(t) - c$
- Arrival intensity is controlled by Markov process (MMRP)
- Interesting issue: Stationary distribution of $X(t)$

Background: Fluid Queues(continued)

- Assume underlying Markov process is birth-death process $Z(t) \in \{0, 1, ..., N\}$
- Set $R(t) = r_i$, when $Z(t)$ is in state i,
- Define partial CDF $P_i(t, x) = P\{X(t) \le x, Z(t) = i\}$
- Consider first how the $P_i(t, x)$ evolves in time

$$\begin{aligned} P_i(t + \Delta t, x) \;&=\; \lambda_{i-1}\Delta t P_{i-1}(t, x - (r_{i-1} - c)\Delta t) + \mu_{i+1}\Delta t P_{i+1}(t, x - (r_{i+1} - c)\Delta t) \\ &+ [1 - (\lambda_i + \mu_i)\Delta t] P_i(t, x - (r_i - c)\Delta t) + O(\Delta t^2) \end{aligned}$$



Background: Fluid Queues(continued)

- Taking the limit $\Delta t \to 0$

$$\frac{\partial}{\partial t}P_i(t,x) + (r_i - c)\frac{\partial}{\partial x}P_i(t,x) \;=\; \lambda_{i-1}P_{i-1}(t,x) + \mu_{i+1}P_{i+1}(t,x) - (\lambda_i + \mu_i)P_i(t,x)$$

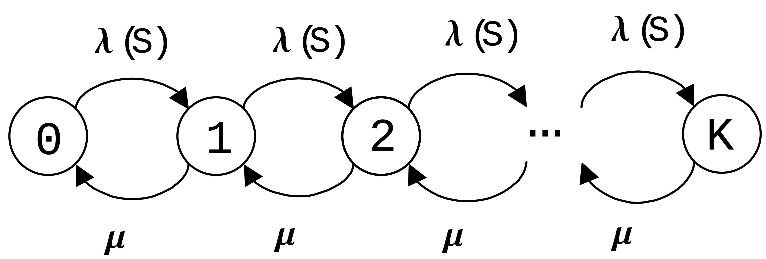
- We are interested in the time-independent properties of process $X(t)$:
  - Setting $\frac{\partial}{\partial t}P_i(t,x) = 0$,
  - Denoting $F_i(x) = P\{X \leq x, Z = i\}$ (equilibrium probabilities)

$$(r_i - c)\frac{\partial}{\partial x}F_i(x) = \lambda_{i-1}F_{i-1}(x) + \mu_{i+1}F_{i+1}(x) - (\lambda_i + \mu_i)F_i(x), \forall i \in \{1, 2, ...N\}$$

- The equations can be expressed in matrix form $\mathbf{M}\frac{d}{dx}\mathbf{F}(x) = \mathbf{A}\mathbf{F}(x)$
- The ODE system is linear, homogenous, constant coefficient system and it is basically easy to solve.

Modelling approach for exponentially averaged queue

- Consider now exponentially averaged queue
  - Instantaneous queue length follows stochastic process $L(t) \in \{0, 1, 2, ..., K\}$
  - Exponentially averaged queue length $S(t)$, $\frac{d}{dt}S(t) = -\alpha(S(t) - L(t))$
  - $L(t)$ is similar to birth-death process
    - Poisson arrivals, packet drop probability depends on $S(t)$,$\lambda(S(t))$
    - Exponential service times with parameter μ



- Interesting issue: Stationary distribution of $S(t)$ and $L(t)$

Modelling approach for exponentially averaged queue (continued)

- Comparison with fluid queue

$Z(t)$ controls the arrival rate	$\sim$	$L(t)$ describes the instantaneous queue length
amount of work in queue $X(t)$	$\sim$	exponentially averaged queue length $S(t)$
$X(t)$ changes with rate $R(t) - c$	$\sim$	$S(t)$ changes with rate $-\alpha(S(t) - L(t))$
$Z(t)$ independent of $X(t)$	$\sim$	$L(t)$ depends on $S(t)$

- Let's take modelling approach similar to fluid queues
  - Consider a process $\{S(t), L(t)\}$
  - Define partial CDF $P_i(t, s) = P\{S(t) \leq s, L(t) = i\}$,
  - and partial PDF $p_i(t, s) = \frac{\partial}{\partial s} P_i(t, s)$

Modelling approach for exponentially averaged queue (continued)

- $P_i(t, s)$ evolves in time step Δt

$$\begin{aligned} P_i(t+\Delta t, s) &= \int_0^{s+\alpha(s-i)\Delta t} [1-(\lambda_i(x)+\mu_i)\Delta t] p_i(t,x)dx \\ &+ \int_0^{s+\alpha(s-(i-1))\Delta t} \lambda_{i-1}(x)\Delta t p_{i-1}(t,x)dx + \\ &+ \int_0^{s+\alpha(s-(i+1))\Delta t} \mu_{i+1}\Delta t p_{i+1}(t,x)dx + O(\Delta t^2) \end{aligned}$$

- Take the limit $\Delta t \to 0$

$$\begin{aligned} \frac{\partial}{\partial t}P_i(t,s) - \alpha(s-i)\frac{\partial}{\partial s}P_i(t,s) &= -\int_0^s (\lambda_i(x)+\mu_i)p_i(t,x)dx \\ &+ \int_0^s \lambda_{i-1}(x)p_{i-1}(t,x)dx + \int_0^s \mu_{i+1}p_{i+1}(t,x)dx \end{aligned}$$

Modelling approach for exponentially averaged queue (continued)

- The time-independent properties of process $x(t)$:
  - Set $\frac{\partial}{\partial t} P_i(t,s) = 0$
  - Define CDF $F_i(s) = P\{S \leq s, L = i\}$ (equilibrium probabilities),
  - and PDF $f_i(s) = \frac{d}{ds} F_i(s)$

$$\alpha(s-i) f_i(s) = \int_0^s (\lambda_i(x) + \mu_i) f_i(x) dx - \int_0^s \lambda_{i-1}(x) f_{i-1}(x) dx - \int_0^s \mu_{i+1} f_{i+1}(x) dx, \forall i$$

- Assuming that $F_i(s)$ has continuous second derivative $\frac{\partial}{\partial s} f_i(s)$, we get

$$\alpha(s-i)\frac{\partial}{\partial s} f_i(s) = (\lambda_i(s) + \mu_i - \alpha) f_i(s) - \lambda_{i-1}(s) f_{i-1}(s) - \mu_{i+1} f_{i+1}(s), \forall i$$

Modelling approach for exponentially averaged queue (continued)

- Thus, we get a system of differential equations

$$\mathbf{M}(s)\frac{d}{ds}\mathbf{f}(s) = \mathbf{A}(s)\mathbf{f}(s),$$

in which

$$\mathbf{f}(s) = \begin{pmatrix} f_0(s) & f_1(s) & \dots & f_K(s) \end{pmatrix}^T$$

$$\mathbf{M}(s) = \alpha \begin{pmatrix} s & & & 0 \\ & s-1 & & \\ & & \ddots & \\ 0 & & & s-K \end{pmatrix}, \mathbf{A}(s) = \begin{pmatrix} \lambda(s)-\alpha & -\mu & & 0 \\ -\lambda(s) & \lambda(s)+\mu-\alpha & -\mu & \\ & & \ddots & \\ 0 & & -\lambda(s) & \mu-\alpha \end{pmatrix}$$

- Process $S(t)$ will not reach boundaries in finite time
$\rightarrow$ boundary conditions: $f_i(0) = 0$, $f_i(K) = 0$, $\forall i$

Remarks about solving the model

- How to solve the system $\mathbf{M}(s)\frac{d}{dx}\mathbf{f}(s) = \mathbf{A}(s)\mathbf{f}(s)$?
- No direct way to solve the DE system analytically
- Numerical solution with Euler methods is difficult
  - $M(s)$ pointwise singular in points $s = \{0, 1, ..., K\}$
    $\longrightarrow$ Boundary conditions do not define the solution uniquely
    $\longrightarrow$ System ”infinitely stiff”, rounding errors dominate the solution
- Other approaches
  - Solve the PDE equations in discretized time
    - ∗ Any initial PDF will approach infinitely close to the stationary PDF
    - ∗ Embedded chain approach
  - Solve the DE system with base function approximations

Examples with M/M/1/K queue

- Consider M/M/1/K queue
  - Arrival intensity λ (independent of $S(t)$)
  - Service intensity μ
- Arrival intensity is now constant and the DE system takes form

$$\mathbf{M}(s)\frac{d}{ds}\mathbf{f}(s) = \mathbf{A}\mathbf{f}(s)$$

- In this case we can derive similar DE system for CDFs $\mathbf{F}(s)$

$$\mathbf{M}(s)\frac{d}{ds}\mathbf{F}(s) = \mathbf{A}'\mathbf{F}(s),$$

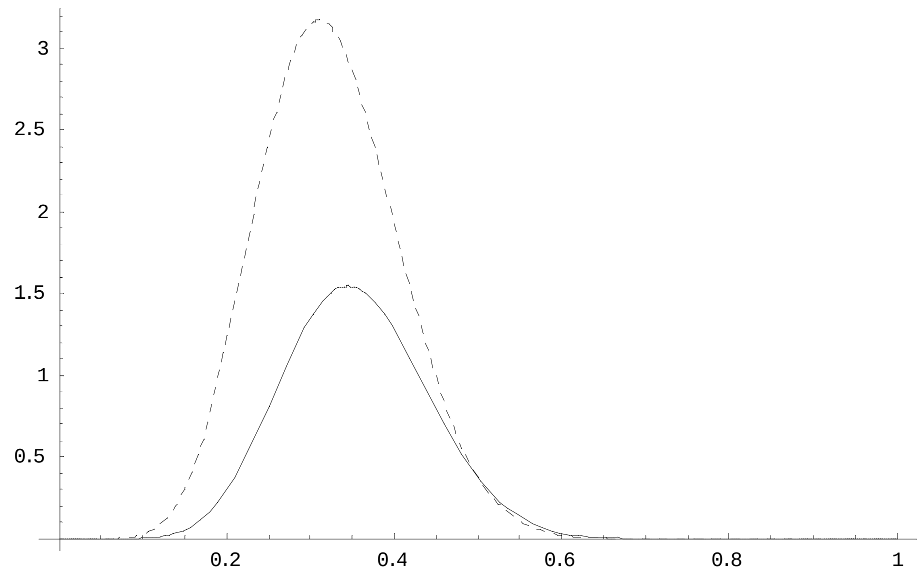
in which $A' = A - \alpha I$

Examples with M/M/1/K queue (continued)

- Analytical solution is found in case K=1

$$\begin{cases} \alpha s \frac{d}{ds} f_0(x) = (\lambda - \alpha) f_0(s) - \mu f_1(s) \\ \alpha(s-1) \frac{d}{ds} f_1(s) = (\mu - \alpha) f_1(s) - \lambda f_0(s) \end{cases}$$

$$\Rightarrow \begin{cases} f_0(s) = s^{\lambda/\alpha - 1}(1-s)^{\mu/\alpha}, s \in [0,1] \\ f_1(s) = s^{\lambda/\alpha}(1-s)^{\mu/\alpha - 1}, s \in [0,1] \end{cases}$$



Examples with M/M/1/K queue (continued)

- $K = 2$, solutions found in the special case $\lambda/\alpha = \mu/\alpha = 2N+1, N \in \mathbf{N}$

$$\begin{cases} \alpha s \frac{d}{ds} f_0(s) &= (\lambda - \alpha) f_0(s) - \mu f_1(s) \\ \alpha(s-1) \frac{d}{ds} f_1(s) &= (\mu + \lambda - \alpha) f_1(s) - \lambda f_0(s) - \mu f_2(s) \\ \alpha(s-2) \frac{d}{ds} f_2(s) &= (\mu - \alpha) f_2(s) - \lambda f_1(s) \end{cases}$$

- Solutions are of the form (The $P_i(s)$ are polynomials of degree $2N$)

$$\Rightarrow \begin{cases} f_0(s) &= s^{\lambda/\alpha-1}(2-s)^{\mu/\alpha+1} P_0(s), s \in [0,2] \\ f_1(s) &= s^{\lambda/\alpha-0}(2-s)^{\mu/\alpha-0} P_1(s), s \in [0,2] \\ f_2(s) &= s^{\lambda/\alpha+1}(2-s)^{\mu/\alpha-1} P_2(s), s \in [0,2] \end{cases}$$

- In the special case $\lambda/\alpha = \mu/\alpha = 2$ solution is also found
  - $f_i(s)$ are picewise polynomials

Further work

- Develop more efficient methods for solving PDE/DE system
  - Speeding up the convergence of time discretized PDE system
  - Embedded chain approach
  - Base function approximations for the DE system
- Verify model results with simulation
- Further development of the model
  - More complex traffic model (e.g. MMRP)
  - Model for two connected queues (AF buffer)