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Modeling the Joint Dynamics of Instantaneous and Exponentially Averaged Queue Lengths

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HELSINKI UNIVERSITY OF TECHNOLOGY **ABSTRACT OF MASTER'S THESIS**

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Differentiated Services architecture is proposed to provide a variety of quality of service levels over the packet switched Internet network. In the DiffServ network complex flow measuring and classification functions are implemented at the network boundary nodes. At the interior nodes the distinctive forwarding behavior between flow aggregates is obtained with simple Per-Hop Behavior (PHB) packet scheduling mechanisms. The effect of the suggested PHB mechanisms and their parameterization on the traffic quality issues has still remained an open question.

The focus of the thesis is on constructing a modeling methodology for the Assured Forwarding scheme that is one of the suggested PHB mechanisms for the DiffServ architecture. The framework for the AF model is adapted from the previous approaches of modeling DiffServ PHB mechanisms. The model aims at capturing the packet level dynamics of the AF buffer in order to evaluate the effect of the AF parameterization on the traffic QoS measures.

The joint dynamics of instantaneous and exponentially averaged queue length involved in the Random Early Detection (RED) congestion control mechanism of the AF buffer is modeled using Markovian fluid queue models. The time evolution of the joint distribution functions of the instantaneous and averaged queue length is governed by Kolmogrov equations that reduce to an ordinary differential equation system in the stationary case.

An analytical solution for the equations is derived in a few special cases, though the general solution is not reached. Solving the ODE system numerically with traditional integration schemes turns out to be unstable. Therefore, two new numerical approaches, method of characteristics and embedded process approach, are constructed. The approaches are based on following the time evolution of the distribution functions until the solution converges to the stationary state. The numerical methods are briefly evaluated and compared. Based on the results both methods are found to give the stationary distributions accurately.

Two extensions for the basic model are briefly illustrated. The Markov Modulated Poisson process is attached to the basic model to capture the behavior of bursty traffic sources. The second extension illustrates how the basic model can be extended to buffers with coupled queues, such as the AF buffer. The numerical methods can be used with slight modifications also with the extended models.

Keywords: Differentiated services, Assured Forwarding, RED, feedback fluid queues

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Eriytettyjen palveluiden (DiffServ) arkkitehtuuri on eräs menetelmistä, joilla voidaan tuottaa eritasoista palvelunlaatua pakettikytekentäisessä Internet-verkossa. DiffServ-verkossa monimutkaiset liikennevuon mittaus- ja luokittelutoiminnot toteutetaan verkon reunasolmuissa. Verkon sisäsolmuissa vuoryhmien kokeman palvelun välille saadaan eroja yksinkertaisten pakettien skedulointiin perustuvien mekanismien (PHB) avulla. Ehdotettujen PHB-mekanismien ja verkon parameterien vaikutusta liikenteen kokemaan laatuun ei kuitenkaan vielä ymmärretä täysin.

Tässä työssä konstruoidaan mallinnustapa taatun välityksen (AF) mekanismille, joka on eräs ehdotetuista DiffServ-verkon PHB- menetelmistä. Ehdotettu AF mallinnustapa perustuu aiempiin DiffServ-verkon mallinnustapoihin. Malli kuvaa AF-puskurin dynamiikka pakettitasolla, ja se tarjoaa mahdollisuuden arvioida puskurin parametrien vaikutusta välitetyn liikenteen kokemaan palvelunlaatuun.

AF-puskurin RED-ruuhkanhallintamekanismin toimintaan liittyvän hetkellisen ja eksponentiaalisesti tasoitetun jononpituuden kytkettyä dynamiikkaa kuvataan Markovisiin nestejonoihin perustuvalla mallilla. Kolmogorovin yhtälöt kuvaavat hetkellisen ja keskiarvoistetun jonopituuden yhteisjakauman ajallista kehitystä. Tasapainotilassa yhtälöt redusoituvat tavalliseksi differentiaaliyhtälösysteemiksi.

Yhtälöiden ratkaisu on esitetty muutamissa erikoistilanteissa. Yleistä ratkaisua ei kuitenkaan ole löydetty, ja numeerinen ratkaiseminen tavallisilla integrointimenetelmillä osoittautuu epästabiiliksi. Kaksi uutta numeerista ratkaisutapaa konstruoidaan, ns. karakterististen käyrien ja upotetun prosessin menetelmä. Menetelmät perustuvat jakaumafunktioiden ajallisen kehityksen seuraamiseen, kunnes ratkaisu konvergoi lopulta tasapainoratkaisuun. Numeerisia menetelmiä testataan ja niiden toimintaa vertaillaan. Testien perusteella kummankin lähestymistavan todetaan antavan tarkkoja ratkaisuja tasapainojakumille.

Lisäksi estitetään lyhesti kaksi laajennusta perusmalliin. Ensimmäinen laajennus on Markov-moduloidun Poisson-prosessin käyttö kuvaamaan purskeisia liikennelähteitä. Lisäksi esitetään malli useamman kytketyn jonon puskurille. Perusmallin ratkaisuun kehitettyjä numeerisia menetelmiä voidaan käyttää pienin muutoksin myös laajennettujen mallien ratkaisuun.

Avainsanat: Eriytetyt palvelut, taattu välitys, RED, takaisinkytketyt nestejonot

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List of Abbreviations

AF	Assured Forwarding
DiffServ	Differentiated Services
EF	Expedited Forwarding
FIFO	First In First Out
IETF	Internet Engineering Task Force
IP	Internet Protocol
ISP	Internet Service Provider
MMPP	Markov Modulated Poisson Process
ODE	Ordinary Differential Equation
pdf	Probability density function
PHB	Per-Hop-Behavior
QoS	Quality of Service
RED	Random Early Detection
SIMA	Simple Integrated Media Access
TCP	Transmission Control Protocol
TD	Threshold Dropping
UDP	User Datagram Protocol

Chapter 1

Introduction

1.1 Background

The current Internet is based on a technology that provides best-effort service without specific performance guarantees for an individual end user. All the data transmitted over the Internet are treated in the same way regardless of the importance aspects the user would like to set for the data. The situation could be compared to a postal service offering only one type of delivery without an option to choose between different urgency or delivery assurance classes. So far, the best-effort service has been adequate for most Internet users. However, both the steadily growing number of Internet users and the introduction of new Internet-based applications have increased the need for Internet service differentiation. A workable method for the service differentiation would also lay a foundation for differentiated pricing of Internet services.

Originally the scope of the Internet Protocol (IP) was to define functions necessary to deliver data packets from a source to a destination over an interconnected system of networks without providing any kind of end-to-end data transmission reliability. Although the Internet Protocol supports quality of service marking in the IP packet, it does not define any policy for service differentiation [1]. Recently research has been carried out to create a simple scheme that would provide a range of service levels in the Internet. The Internet Engineering Task Force (IETF) has suggested several approaches for the implementation of Quality of Service (QoS) in the Internet, though none of the approaches has yet been widely approved as the basis of the future Internet.

The most notable proposals for the Internet QoS mechanism are the Integrated Service/Resource Reservation Protocol method (IntServ/RSVP) suggested in [2] and the Differentiated Services (DiffServ) architecture presented in [3]. The IntServ/RSVP introduced a service differentiation mechanism that is based on resource reservation and per flow scheduling inside the network. Due to excessive computational and memory requirements needed to control a large amount of data flows at the high speed links, the IntServ/RSVP method is poorly scalable to larger networks [4]. In the DiffServ architecture the scalability problem is solved by setting the sophisticated flow classification operations at the edges of the network and implementing the QoS inside the network with simple aggregate Per-Hop Behavior (PHB) mechanisms. So far the DiffServ architecture is considered to be a more promising alternative for the implementation of QoS in the Internet and, in fact, it is already being deployed to some extent.

1.2 Motivation and aim of the thesis

The effect of the proposed DiffServ PHB mechanisms on the network performance is still under research. Quality of service parameters in the DiffServ network, e.g., packet loss probability, delay and fairness in bandwidth allocation, are essential measures for evaluating the DiffServ PHB mechanisms. This information can also be utilized by the Internet Service Providers (ISP) for network dimensioning and design purposes.

In this thesis we focus on modeling the packet level dynamics of the Assured Forwarding (AF) PHB buffer. The emphasis is on constructing an analytical model of an AF PHB buffer that could be used to evaluate the effect of the buffer parameters on the traffic QoS measures, such as expected delay and packet loss probability. It is suggested that Random Early Detection (RED) mechanism or its variations would be utilized in the AF buffer congestion control mechanisms. Thus, the focus is put on the modeling of the RED mechanism as a part of an AF buffer.

The RED involves a packet acceptance mechanism based on the exponentially averaged buffer level. This leads us to consider the joint dynamics of the instantaneous and related exponentially averaged queue lengths. With certain assumptions about the traffic flows, the joint dynamics can be captured with a special class of Markovian feedback fluid queue model.

1.3 Structure of the thesis

The structure of the thesis is the following. First in Chapter 2, the DiffServ architecture and especially the Assured Forwarding PHB group are described. In Chapter 3, previous approaches to model the DiffServ network and the RED mechanism are briefly introduced. The RED algorithm is described in more detail in Appendix A. Based on the previous work, a framework for modeling the AF buffer with RED congestion control is presented. In Chapter 4, a mathematical model is constructed for the AF buffer mechanisms involving joint dynamics of the instantaneous and averaged queue lengths. The model is based on the Markovian feedback fluid queues and it provides a set of differential equations governing the stationary probability distribution functions for the joint system of the instantaneous and exponentially averaged queue lengths. Analytical solutions for the probability distribution functions in a few special cases are derived in Chapter 5. Due to the problems in finding the general solution, two numerical approaches for solving the model are constructed and evaluated in Chapter 6. A numerical example about the stationary properties of a buffer system with RED mechanism is also presented in Chapter 7. In Chapter 8 two extensions to the basic model are discussed. The first extension introduces a more complex traffic model for bursty traffic sources. The second extension shows how the single queue buffer model can be extended to a system of coupled queues as in the AF buffer. Finally, some conclusions are drawn in Chapter 9.

Chapter 2

Differentiated Services Architecture

2.1 Introduction

In this chapter we give an overview of the Differentiated Services (DiffServ) network architecture and its basic concepts and ideas. One of the suggested methods for the packet forwarding in the DiffServ network, Assured Forwarding PHB group, is introduced in more detail. In the rest of the thesis the focus is specifically on the modeling of the AF packet forwarding behavior in the DiffServ network router.

2.2 Structure of the DiffServ network

The Differentiated Services architecture proposed in [3] defines a framework for implementing scalable service differentiation in the Internet. The DiffServ architecture is composed of several mechanisms implemented in the network nodes. These mechanisms include traffic conditioning functions, packet classification functions and a set of per-hop forwarding behaviors. The scalability of the DiffServ architecture is achieved by implementing complex flow level classification and conditioning functions only at the boundary network nodes and applying simple packet level per-hop behaviors on traffic aggregates at the interior network nodes.

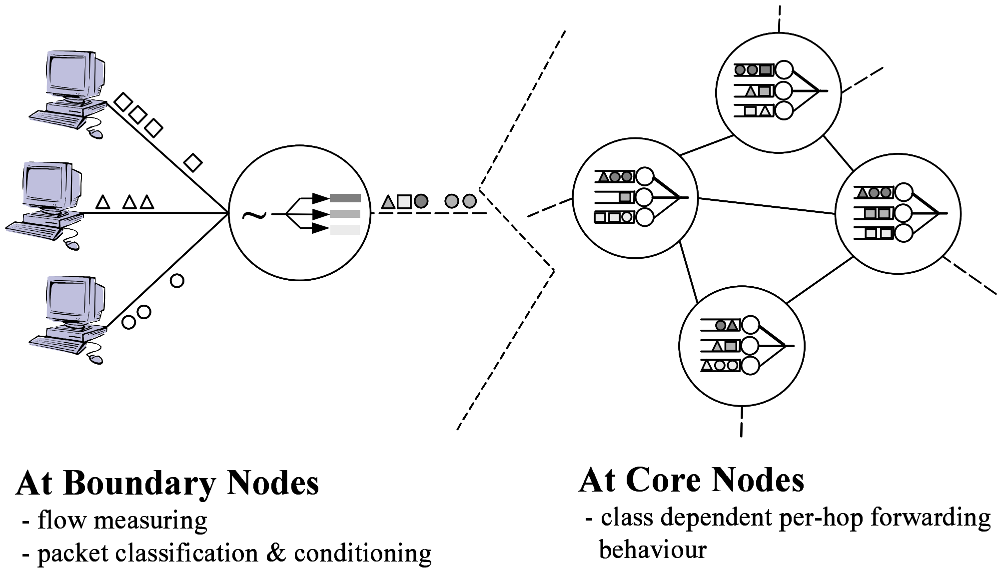


Figure 2.1: Illustration of the DiffServ architecture and its main functions.

The top level DiffServ architecture is illustrated in Figure 2.1. The DiffServ network consists of two, functionally distinctive types of routers or network nodes, namely, the network boundary nodes, to which end-user terminals are connected, and the core nodes that forward traffic in the interior parts of a DiffServ network.

At the boundary nodes the temporal properties of the traffic flow are measured and compared against a predefined traffic profile agreed between the network user and the network service provider. Packets are classified into the Per-Hop Behavior classes based on this flow level comparison. In this phase the packets may be delayed or dropped to shape traffic flows to bring them in compliance with the corresponding traffic profiles.

In the interior parts of the network the packets are not traced anymore on the flow level. Packets that are marked into the same PHB class will experience similar forwarding behavior in the core nodes independent of the actual properties of the flow they belong to. Per-hop forwarding behavior in the core nodes is implemented by means of buffer management and packet scheduling mechanisms. The observable packet forwarding behavior will often depend heavily on the relative load on the link. However, when several PHB aggregates are competing for the buffer and bandwidth resources with different priorities, useful behavioral distinctions between the PHB classes may be achieved.

2.3 Per-Hop Behavior groups

The DiffServ architecture defines general Per-Hop Behavior group specification guidelines, but leaves the detailed implementation, i.e., packet classification and metering functions and PHB classes open for suggestions. Several PHB group approaches have been introduced, e.g., Assured Forwarding (AF) in [5] and Expedited Forwarding (EF) in [6]. In the following the AF PHB group is described in more detail as the further analysis in this report is mainly motivated by the AF PHB group.

2.3.1 Assured Forwarding PHB group

Assured Forwarding PHB proposal defines a group of four independently forwarded PHB classes. Within each AF class, packets can be assigned to one of three different levels of drop precedence. The structure of the AF PHB group is shown in Figure 2.2. In case of long-term congestion, the drop precedence level defines the relative importance of the packet within the AF class. High drop precedence means that a packet will be discarded more preferably compared to a packet with low drop precedence.

Specification guidelines

The AF PHB group definition in [5] contains the following specifications, which an AF PHB compatible DiffServ node must conform exactly for full compliance. Specifications define requirements for packet forwarding behavior and mutual relation of AF classes and drop precedence levels.

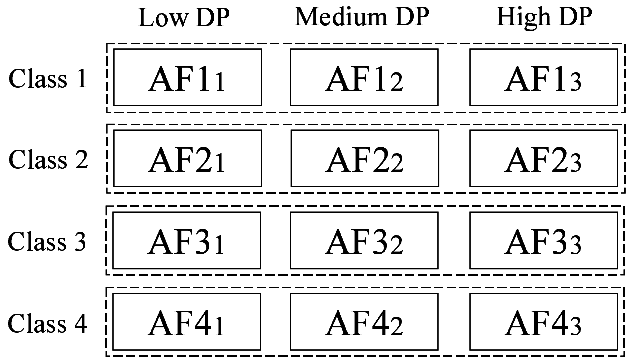


Figure 2.2: Structure of the AF PHB Group.

- *Packets in one AF class must be forwarded independently from packets in another AF class.*
- *A DiffServ node must not aggregate two or more AF classes together.*
- *A DiffServ node must allocate a configurable, minimum amount of forwarding resources (buffer size and bandwidth) to each implemented AF class.*
- *An AF implementation must specify how the excess resources are allocated between AF classes.*
- *Within an AF class, a DiffServ node must not forward an IP packet with smaller probability if it contains a drop precedence value p than if it contains a drop precedence value q when* $p < q$.
- *Within each AF class, a DiffServ node must accept all three drop precedence codepoints and they must yield at least two different levels of loss probability. If only two different drop precedence levels are implemented the codepoints AFx2 and AFx3 shall be combined to the higher drop precedence level.*
- *A DiffServ node must not reorder AF packets of the same microflow when they belong to the same AF class regardless of their drop precedence.*
- *An AF implementation must attempt to minimize long-term congestion within each class by dropping packets, while handling short-term congestion by queueing packets.*

The AF PHB can be implemented with a simple buffer mechanism. Each AF class has its own separate buffer with a predefined buffer size and bandwidth allocation. Packets are divided into the buffers based on the AF class mark in the packet. The buffering mechanism will handle short-term congestion, i.e., packet bursts, by queueing packets into the buffer. Long-term congestion will be controlled by dropping the incoming packets. The idea in this is to give an equal treatment to the flows that have different short-term properties but similar long-term characteristics. The dropping algorithm should compute smoothed buffer level to track long-term congestion properties of the buffer.

This smoothed buffer level is then used in determining when packets should be discarded.

One suggested approach in the implementation of the dropping algorithm is to use hard thresholds. Each drop precedence has a certain congestion level threshold. The packets aggregated to this drop precedence will be dropped, if the smoothed buffer level exceeds the threshold value.

Random Early Detection (RED) mechanism [7] can also be used in the dropping algorithm. The RED mechanism was invented to reduce transfer rate oscillation of TCP flows and thus to improve the throughput of TCP flows on congested nodes. The RED mechanism starts to drop incoming packets randomly with a certain probability when the congestion level reaches the threshold th_{min}. The dropping probability will increase linearly from zero to p_{max} until the congestion level limit th_{max} is reached, after which all the packets are dropped. The dropping probability function is illustrated in Figure 2.3. A more detailed description about the RED algorithm can be found in Appendix A.

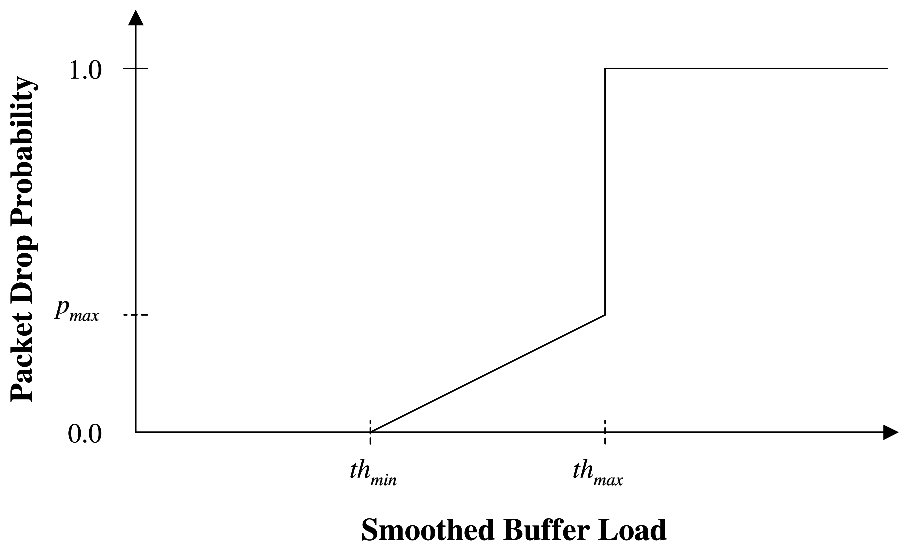


Figure 2.3: Illustration of the RED packet dropping function.

The drop precedence levels in the AF buffer may be implemented with hard thresholds, RED regions or combinations of these two. An example of the AF PHB buffer with four queues and combination of hard threshold and RED dropping mechanism is illustrated in Figure 2.4.

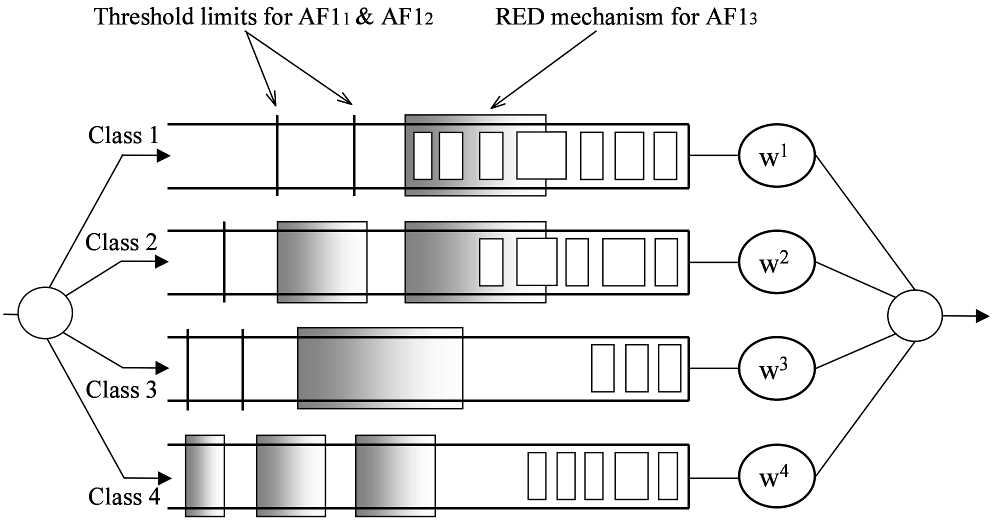


Figure 2.4: Example of AF PHB Implementation (Adapted from [8]).

Implementation issues

The AF PHB definition in [5] does not specify any detailed parameterization for the PHB classes nor a mechanism for the packet classification and drop precedence setting in the boundary nodes. Some approaches have been suggested, e.g., an Olympic Service mechanism. For clarifying the actual implementation of the AF mechanism, we consider the Olympic Service approach briefly.

The olympic service consists of three service classes, bronze, silver, and gold. Packets assigned to the gold class would experience lighter load than the packets assigned to the silver class. A similar kind of relationship would exist between the silver and bronze classes. The olympic classes could be mapped to AF classes 1, 2 and 3. The differences between the classes could be achieved, e.g., by allocating more bandwidth for the gold packet class than for the bronze class. Inside a class, packets could be further separated by giving them either low, medium, or high drop precedence. The drop precedence marking could be used in controlling whether the traffic flow is in accordance with the traffic profile agreed between the user and network service provider. The drop precedence can be assigned to a packet with, e.g., a leaky bucket scheme. In the leaky bucket mechanism packets are buffered in a virtual queue that is served with a predefined service rate. The drop precedence is assigned to arriving packets based on the occupancy level of the virtual queue. For example, if the queue occupancy level is constantly high, it implies that the traffic flow violates the traffic profile and the packets are marked to the high drop precedence.

Chapter 3

Modeling DiffServ Mechanisms

3.1 Introduction

In this chapter some previous approaches to model and analyze the performance aspects of the DifferServ network are briefly introduced. Because RED is closely involved with the AF buffer congestion control mechanism, we also illustrate suggested ways to model the RED behavior. Finally we describe an approach for modeling the AF PHB mechanism in order to provide a tool to evaluate its performance and fairness issues.

3.2 DiffServ models

DiffServ Architecture performance issues have been widely studied both with simulation and analytical methods. Most of the analytical models are based on evaluating the traffic quality over a single, congested network node. The models provide information about the QoS measures of the forwarded traffic, such as packet dropping probabilities and forwarding delays experienced by the traffic flows.

Buffer model of AF PHB

In [9] a model is constructed for an AF buffer with the Threshold Dropping (TD) mechanism. The TD mechanism controls the incoming traffic based on the instantaneous buffer load. In the paper the buffer is analyzed with a

Markov chain model by assuming Poisson flows and exponentially distributed packet service times. The model was utilized in determining the QoS requirements for the packet loss and expected forwarding delay.

SIMA PHB with TCP and UDP sources

In [10] an approach for analyzing the DiffServ network with aggregates of TCP and UDP flows is illustrated. In the paper a DiffServ node with SIMA PHB is considered. Again, assuming that the traffic is generated from Poisson sources and the packet service times are exponentially distributed, the packet level dynamics of the SIMA buffer is described with a Markov model. The behavior of the TCP congestion control mechanism is included into the buffer model using the relation between the packet dropping probability in the buffer and the average sending rate of the aggregated TCP flows. This modeling approach provides means to analyze the fairness of bandwidth division between TCP and UDP flows over a single DiffServ network node and evaluate the effect of the buffer parameterization on the traffic QoS measures.

3.3 RED models

The RED mechanism has been widely studied since its introduction. However, the emphasis has been on modeling its functioning with congestion sensitive TCP traffic instead of as a part of QoS mechanisms. Some of the previous approaches for RED modeling are briefly discussed in the following.

RED stability issues

The RED-TCP dynamics is studied, e.g., in [11]. The paper describes a performance analysis of RED congestion control algorithm with heterogenous TCP flows by means of a simple stochastic differential equation formulation. Another approach is presented in [12], in which the stability of the RED mechanism with aggregates of TCP flows is evaluated using an ordinary differential equation model. Both of the approaches provide suggestions for the RED parameterization in order to obtain stable load into the RED buffer with congestion sensitive TCP flows.

Performance measures of a RED buffer

Another approach for evaluating the dynamics of a RED buffer is considered in [13]. A Markovian model for a RED buffer with Poissonian traffic flows is constructed. The model is based on discretization of the continuous state variable of the averaged buffer length. The model provides information about the dropping probability and expected delay caused by the RED buffer, however, contrary to [11] or [12] it does not provide means to analyze the stability issues of the RED and TCP flows.

3.4 An approach for modeling an AF PHB

From the QoS point of view an applicable AF PHB model should provide information about the effect of the buffer parameters to the traffic performance measures, such as expected forwarding delay or packet dropping probability. In addition, the model should provide information about the fairness issues, e.g., the bandwidth sharing between the congestion sensitive TCP flows and greedy UDP flows forwarded over an AF buffer.

In order to model the QoS properties of an AF buffer the ideas presented in the previous modeling approaches may be utilized. The packet level dynamics of AF PHB buffer with RED packet dropping mechanism can be captured with a Markovian model by assuming Poissonian traffic sources such as illustrated in [13]. Such a model would provide information about the delay and dropping probability associated with different AF classes, though, it lacks to capture the stability issues of the AF buffer with TCP sources. However, the packet level buffer model could be utilized in a similar manner as in [10] to evaluate the behavior of an AF buffer with aggregates of TCP flows.

In the rest of the thesis we continue to process the above idea of using a Markovian model for the packet level dynamics of the AF PHB buffer.

Chapter 4

Model for Instantaneous and Averaged Queue Lengths

4.1 Introduction

In this chapter a model for an AF buffer queue with the RED congestion control mechanism is constructed. The model aims to capture the essential dynamics of the RED packet dropping mechanism based on the exponentially averaged queue length. For tractability of the problem we consider a case in which the averaging is done continuously instead of momentarily at the packet arrivals. However, from the AF modeling point of view this approach is not essential, because the definition of the AF requires a control mechanism based on the smoothed buffer load.

4.2 Mathematical model

Consider a queueing system with a single buffer. Assume that the buffer has K system places, i.e., one server and $K - 1$ waiting positions. It is assumed that customers arrive to the system according to a Poisson process with a constant arrival rate λ. Customer service times are exponentially distributed with parameter μ. If the queue is empty when the customer is accepted to the system, the customer will be served instantly. Otherwise it is stored into a FIFO queue to wait for the service. The queueing system contains an acceptance mechanism that rejects an arriving customer with a certain probability that depends on the exponentially averaged queue length. Without the acceptance mechanism, the queueing system would be equivalent to the traditional $M/M/1/K$ queue.

Averaging process

Denote the instantaneous queue length at time t with $L(t)$. The exponentially averaged queue length $S(t)$ at time t is defined by

$$S(t) = \int_0^{\infty} L(t-u)\alpha e^{-\alpha u} du, \tag{4.1}$$

in which α is an averaging constant. The instantaneous value of process $S(t)$ at certain instant t is the weighted average of the values of $L(t)$ prior to the instant t. Value $L(t-u)$ is weighted with an exponentially decaying factor $\alpha e^{-\alpha u}$. It can be seen that between the changes of process $L(t)$ the process $S(t)$ obeys the differential equation

$$\frac{d}{dt}S(t) = -\alpha(S(t) - L(t)), \tag{4.2}$$

i.e., the rate at which $S(t)$ changes is proportional to the difference at time t between the instantaneous and exponentially averaged queue length.

The exponentially averaged queue length process $S(t)$ differs from the averaging process defined in the RED mechanism (see, Appendix A). The RED averaging algorithm computes the exponentially weighted moving average of the queue length and the averaged queue length is updated momentarily at the packet arrival instants. However, both these averaging processes weights the past values of the instantaneous queue length with exponentially decreasing factor. Thus, the exponentially averaged queue length process $S(t)$ can be considered as a good approximation of the RED averaging process.

Customer acceptance mechanism

The customer acceptance mechanism is based on the exponentially averaged queue length $S(t)$. A customer that arrives to the system at time t is always rejected if the buffer is full, i.e., if $L(t) = K$. In addition, a customer may be rejected with a certain dropping probability that depends on the value of $S(t)$. Let $p(x), p(x) \in [0,1]$, denote the dropping probability associated with exponentially averaged queue length $S(t) = x$.

4.2.1 Structure of the queue model

The instantaneous queue length $L(t)$ constitutes a stochastic process with a finite state space $\mathcal{N} = \{i | i = \{0, 1, ..., K\}\}$. The state transitions for $L(t)$ occur only between the neighboring states, i.e., from state i to $i+1$ or from state i to $i-1$. Based on the assumption about the traffic sources, customers arrive to the queueing system according to the Poisson process with intensity λ. However, the arriving customer may be rejected with a dropping probability that depends on the current sate of the process $S(t)$. Thus, the state transition rate to the upper state depends on the exponentially averaged queue length process $S(t)$. For convenience, it is simply denoted that the arrival intensity depends on the current value of process $S(t) = x$, such that $\lambda(x) = (1-p(x))\lambda$. The customer service intensity is constant with intensity μ, i.e. the state transition rate to the lower state is constant. The state transition diagram for the process $L(t)$ is shown in Figure 4.1.

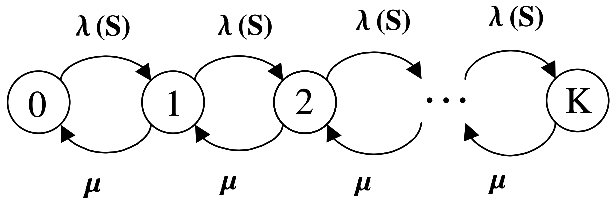


Figure 4.1: State transition diagram of instantaneous queue length process.

The exponentially averaged queue length process $S(t)$ behaves deterministically between the transitions of the process $L(t)$ according to equation (4.2). The interaction between the processes $L(t)$ and $S(t)$ is illustrated in Figure 4.2.

The system resembles the situation in classical Markov modulated fluid queues [14], where a continuous-state process is regulated by a discrete-state Markov

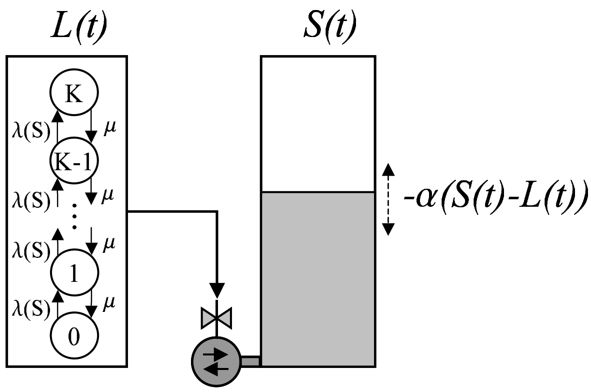


Figure 4.2: Interaction between the instantaneous and exponentially averaged queue lengths.

process. In this case the system is more complex because the rate at which process $S(t)$ changes at time t not only depends on $L(t)$ but also on $S(t)$ itself. In addition, there is a feedback mechanism from $S(t)$ that controls the transition rates of $L(t)$. The system has similarities with feedback fluid queues that are analyzed by Scheinhardt in [15]. The study focuses on analyzing buffer models in which the feedback alternates in a piecewise manner. The buffer model in this case is slightly different due to the continuously changing feedback.

The processes $L(t)$ or $S(t)$ alone are not Markov processes due to the interdependence between them. Consider the two-dimensional, joint process $(L(t), S(t))$ with state space

$$\mathcal{S}_{LS} = \{(i, x) | i = \{0, \ldots, K\}, x \in [0, K]\}.$$

Because of the Poissonian traffic source and the random packet dropping mechanism connected with the state of the process $S(t)$, the state transition probabilities depend only on the current state of the process $(L(t), S(t))$. Thus the process $(L(t), S(t))$ fulfills the Markov property.

4.2.2 Time evolution of the distribution functions

In order to evaluate the behavior of the joint process $(L(t), S(t))$ we consider the partial cumulative distribution function of the process $(L(t), S(t))$,

$$F_i(t,x) = P\{L(t) = i, S(t) < x\}, \quad i \in \mathcal{N}. \tag{4.3}$$

Correspondingly we can define the partial probability density function

$$f_i(t,x) = \frac{\partial}{\partial s} F_i(t,x), \quad i \in \mathcal{N}. \tag{4.4}$$

Kolmogorov equations

The time evolution of the partial probability density functions $f_i(t,x)$ are governed by Kolmogrov forward equations (see, e.g., [16]), which in this case can be written

$$\begin{aligned} \frac{\partial}{\partial t} f_i(t,x) - \alpha(x-i)\frac{\partial}{\partial x} f_i(t,x) &= \\ -[\lambda_i(x) + \mu_i - \alpha] f_i(t,x) + \lambda_{i-1}(x) f_{i-1}(t,x) &+ \mu_{i+1} f_{i+1}(t,x), \quad i \in \mathcal{N}, \end{aligned} \tag{4.5}$$

in which

$$\lambda_i(x) = \begin{cases} \lambda(x), & i \in \mathcal{N}/\{K\}, \\ 0, & i = K, \end{cases} \qquad \mu_i = \begin{cases} \mu, & i \in \mathcal{N}/\{0\}, \\ 0, & i = 0, \end{cases}$$

and

$$f_i(t,x) \equiv 0, \quad i \notin \mathcal{N}.$$

An exact proof of the Kolmogrov equations (4.5) is omitted, however, we may reason the equations in a rather intuitive manner.

Consider the evolution of the process $(L(t), S(t))$ over a small time step Δt. Assume, that the system is in state $(L(t+\Delta t) = i, S(t+\Delta t) = x)$ at some instant $t + \Delta t$. At the instant t the instantaneous queue length process $L(t)$ may have been in one of the states $L(t) = i$, $L(t) = i-1$ or $L(t) = i+1$ depending on the the state transitions that occur within the time step Δt. The probabilities of the single state transitions are in proportion to $O(\Delta t)$.

Multiple state transitions may also occur, however, the transition probabilities are in proportion to $O(\Delta t^2)$.

The value of exponentially averaged queue length at the time instant t is $S(t) = x - \Delta x$. The factor Δx is determined by the differential equation (4.2) and the state of process $L(t)$ at time t. If the system does not change the state during the time step Δt then $\Delta x = -\alpha(x - i)\Delta t$. Otherwise, the factor $\Delta x \approx -\alpha(x - (i+1))\Delta t$ or $\Delta x \approx -\alpha(x - (i-1))\Delta t$ if $L(t) = i + 1$ or $L(t) = i - 1$ respectively.

The evolution of the partial cumulative distribution function $F_i(t+\Delta t, x)$ over a time step Δt can be thus described by

$$\begin{aligned} F_i(t+\Delta t, x) &= \int_0^{x+\alpha(x-i)\Delta t} [1 - (\lambda_i(s) + \mu_i)\Delta t] f_i(t, s) ds \\ &+ \int_0^{x+\alpha(x-(i-1))\Delta t} \lambda_{i-1}(s)\Delta t f_{i-1}(t, s) ds \\ &+ \int_0^{x+\alpha(x-(i+1))\Delta t} \mu_{i+1}\Delta t f_{i+1}(t, s) ds + O(\Delta t^2), \quad i \in \mathcal{N}, \end{aligned} \tag{4.6}$$

in which

$$\lambda_i(x) = \begin{cases} \lambda(x), & i \in \mathcal{N}/\{K\}, \\ 0, & i = K, \end{cases}, \quad \mu_i = \begin{cases} \mu, & i \in \mathcal{N}/\{0\}, \\ 0, & i = 0. \end{cases}$$

The first integral term in equation (4.6) describes the value of cumulative probability function at time instant t, i.e., $F_i(t, x + \alpha(x - i))$, weighted with a probability $1 - (\lambda_i(x) + \mu_i)\Delta t$ that the process remains in state $L(t) = i$ during the time step Δt. The second and the third integral term correspond the cumulative probability distribution function values from the neighboring states $i+1$ and $i-1$, respectively, weighted with the probability that the $L(t)$ changes to the state i during the time step Δt. The terms involving higher order terms of Δt, i.e. the multiple jumps, are denoted simply with $O(\Delta t^2)$.

By rearranging the terms of equation (4.6) and taking the limit $\Delta t \to 0$, we get a system of partial differential equations that constitute the forward Kolmogorov equations for the process $(L(t), S(t))$,

(4.7)

$$\begin{aligned}\frac{\partial}{\partial t}F_i(t,x) - \alpha(x-i)\frac{\partial}{\partial x}F_i(t,x) &= -\int_0^x (\lambda_i(s)+\mu_i)f_i(t,s)ds \\ &+ \int_0^x \lambda_{i-1}(s)f_{i-1}(t,s)ds \\ &+ \int_0^x \mu_{i+1}f_{i+1}(t,s)ds, \quad i \in \mathcal{N}.\end{aligned} \tag{4.8}$$

Assuming that $F_i(t,x)$ has continuous second derivative $\frac{\partial}{\partial x}f_i(t,x)$ and differentiating equation (4.8) with respect to x, we get the system of partial differential equations in (4.5), i.e., the Kolmogrov equations.

Matrix representation of Kolmogorov equations

The Kolmogrov equations (4.5) can be written more conveniently in matrix form,

$$\frac{\partial}{\partial t}\mathbf{f}(t,x) + \mathbf{D}(x)\frac{\partial}{\partial x}\mathbf{f}(t,x) = \mathbf{Q}^{\mathrm{T}}(x)\mathbf{f}(t,x), \tag{4.9}$$

in which

$$\begin{aligned}\mathbf{f}(t,x) &= (f_0(t,x), f_1(t,x), ..., f_K(t,x))^T, \\ \mathbf{D}(x) &= \alpha\ \mathbf{diag}(-x, 1-x, 2-x, ..., K-x), \\ \mathbf{Q}(x) &= \begin{pmatrix} (\alpha-\lambda(x)) & \lambda(x) & 0 & \cdots & 0 \\ \mu & (\alpha-\lambda(x)-\mu) & \lambda(x) & \cdots & \vdots \\ 0 & \mu & (\alpha-\lambda(x)-\mu) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \lambda(x) \\ 0 & \cdots & 0 & \mu & (\alpha-\mu)\end{pmatrix}.\end{aligned}$$

4.2.3 Stationary equations

Due to the finite nature of the buffer system, the time dependent distribution of process $(L(t), S(t))$ will approach a stationary distribution as $t \to \infty$. Denote the partial probability density functions in stationary case with

$$\mathbf{f}(x) = \lim_{t\to\infty}\mathbf{f}(t,x).$$

In the equilibrium the pdf functions do not evolve in time anymore, i.e., $\frac{d}{dt}\mathbf{f}(t,x) \equiv \mathbf{0}$, Applying this onto equation (4.9) we get a system of ordinary differential equations (ODE) describing the stationary properties of the partial probability density functions,

$$\mathbf{D}(x)\frac{d}{dx}\mathbf{f}(x) = \mathbf{Q}^{\mathrm{T}}(x)\mathbf{f}(x). \tag{4.10}$$

4.2.4 Remarks about the boundary conditions

The ODE system (4.10) governs the behavior of the partial probability density functions $\mathbf{f}(x)$ for the joint system of instantaneous and exponentially averaged queue lengths. Some remarks about the boundary conditions for the density functions can be made. However, due to the structure of the ODE system they do not fully determine the solution.

Behavior of the averaging process

Consider the exponentially averaged queue length process $S(t)$ in more detail. If $L(t) = i$ for a certain time interval $t \in (t_1, t_2)$, the solution of the equation (4.2) is

$$S(t) = i + (S(t_1) - i)e^{-\alpha(t-t_1)}, t \in (t_1, t_2), \tag{4.11}$$

i.e., the difference between $S(t)$ and instantaneous queue length $L(t)$ decreases exponentially. If the process $L(t)$ remains only a finite time in the boundary states $L(t) = 0$ or $L(t) = K$, the process $S(t)$ will never reach these boundary values nor pass them. Say, $S(0) \in (0, K)$, then from the previous reasoning we instantly get $S(t) \in (0, K)$ for $t \geq 0$.

Thus, for the partial probability density functions we get the boundary values

$$f_i(x) = 0, \quad x < 0, \quad x > K, \quad i \in \mathcal{N}. \tag{4.12}$$

Similarly for the partial cumulative distribution functions we get

$$F_i(x) = 0, \quad x \leq 0, \quad i \in \mathcal{N}. \tag{4.13}$$

It is worth to note that for $x \geq K$, $F_i(x)$ will equal to the stationary probability of instantaneous queue length process being in state i,

$$F_i(K) = \int_0^K f_i(x)dx = \lim_{t\to\infty} P\{L(t) = i\} = \pi_i, \quad i \in \mathcal{N}.$$

However, in the general case the stationary probabilities π_i are unknown.

Remarks about singular $\mathbf{D}(x)$

Due to the pointwise singularity of $\mathbf{D}(x)$ at the integer points $x = 0, 1, \ldots, K$, the differential equation governing the behavior of $\frac{d}{dx} f_i(x)$ is not determined at point $x = i$. This causes some problems, such that the boundary conditions described above would not fully determine the solution for the system. More research needs to be done to fully understand the exact behavior of the ODE system and its implications to its solutions.

Chapter 5

Analytical Solutions for Equilibrium Distributions

5.1 Introduction

General solution for the stationary equations

$$\mathbf{D}(x)\frac{d}{dx}\mathbf{f}(x) = \mathbf{Q}^{\mathrm{T}}(x)\mathbf{f}(x),$$

is difficult to find. However, if we consider the exponentially averaged queue length process driven by an ordinary $M/M/1/K$ system, i.e., there is no RED feedback mechanism involved, the differential equations transform into more easily solvable form. The analytical solutions are presented for the two state system $K = 1$ and in a symmetric case for the three state system $K = 2$. Though the solutions do not have much practical use, they may give information about the qualitative behavior of the solutions as well as about the structure of the general solution for the equations.

5.2 Solution for M/M/1/1 system

Consider an ordinary $M/M/1/1$ queueing system and related exponentially averaged queue length process with no RED mechanism involved. The arrival intensity λ is not dependent on the current state of the exponentially averaged queue length variable x, i.e. λ is constant and there is no feedback dynamics involved.

Stationary equations for the two state system

In the two state system the partial probability density functions $f_0(x)$ and $f_1(x)$ are governed by ODE system,

$$\begin{cases} \alpha(x-0)\frac{d}{dx}f_0(x) &= (\lambda-\alpha)f_0(x)-\mu f_1(x), \quad 0 \le x \le 1, \\ \alpha(x-1)\frac{d}{dx}f_1(x) &= (\mu-\alpha)f_1(x)-\lambda f_0(x), \quad 0 \le x \le 1. \end{cases} \tag{5.1}$$

Boundary conditions for the differential equations are given by

$$f_i(x) = 0, \qquad x \le 0, \quad i = 0, 1, \tag{5.2}$$

$$F_i(x) = \int_0^x f_i(x)dx = \pi_i, \quad x \ge 1, \quad i = 0, 1, \tag{5.3}$$

where π_i denotes the stationary probability of having i customers in an $M/M/1/1$ queue. For an $M/M/1/K$ queue the stationary probabilities π_i are equal to

$$\pi_i = \frac{\rho^i}{\sum_{j=0}^{K} \rho^j} = \frac{1-\rho}{1-\rho^{K+1}}\,\rho^i, \qquad i = 0, 1, \ldots, K, \tag{5.4}$$

in which $\rho = \lambda/\mu$.

General solution

The beta distribution with unknown parameters $c_i, p_i, q_i, i = 0, 1$,

$$\begin{cases} f_0(x) &= c_0 x^{p_0}(1-x)^{q_0}, \quad 0 \le x \le 1, \\ f_1(x) &= c_1 x^{p_1}(1-x)^{q_1}, \quad 0 \le x \le 1, \end{cases}$$

is used as a trial solution for the differential equation system (5.1). Requiring that the solution satisfy the differential equations (5.1) and the boundary conditions (5.2) and (5.3), we can determine the unknown parameters of the trial solution,

$$\begin{cases} f_0(x) &= x^{\frac{\lambda}{\alpha}-1}(1-x)^{\frac{\mu}{\alpha}}/B(\frac{\lambda}{\alpha},\frac{\mu}{\alpha}), \quad 0 < x \le 1, \\ f_1(x) &= x^{\frac{\lambda}{\alpha}}(1-x)^{\frac{\mu}{\alpha}-1}/B(\frac{\lambda}{\alpha},\frac{\mu}{\alpha}), \quad 0 \le x < 1, \end{cases} \tag{5.5}$$

where $B(\cdot,\cdot)$ is the beta function defined by

$$B(z_1, z_2) = \int_0^1 y^{z_1-1}(1-y)^{z_2-1}\,dy.$$

Qualitative behavior of the solution

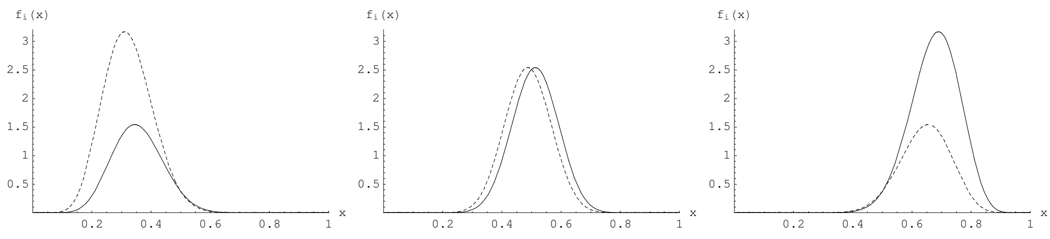
The qualitative properties of the solution for the two state system (5.5) are considered. The ratios λ/α and μ/α affect the qualitative behavior of the probability density functions in the boundaries of the their definition area, $x \in [0,1]$. With $\lambda/\alpha = 1$ there is a change in the limit value $\lim_{x\to 0} f_0(x)$ and similarly, with $\mu/\alpha = 1$ for the limit $\lim_{x\to 1} f_1(x)$, such that

$$\lim_{x\to 0^+} f_0(x) \quad = \quad \begin{cases} 0, & \lambda/\alpha > 1, \\ 1/B(\frac{\lambda}{\alpha}, \frac{\mu}{\alpha}), & \lambda/\alpha = 1, \\ \infty, & \lambda/\alpha < 1, \end{cases} \tag{5.6}$$

$$\lim_{x\to 1^-} f_1(x) \quad = \quad \begin{cases} 0, & \mu/\alpha > 1, \\ 1/B(\frac{\lambda}{\alpha}, \frac{\mu}{\alpha}), & \mu/\alpha = 1, \\ \infty, & \mu/\alpha < 1. \end{cases} \tag{5.7}$$

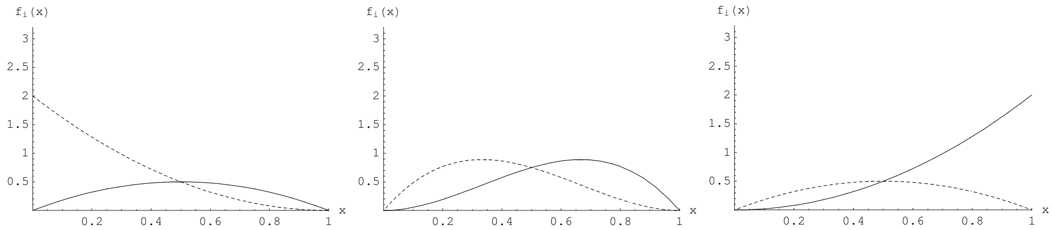
The threefold behavior of the solutions at the boundaries is illustrated in Figures 5.1, 5.2 and 5.3. Figure 5.1 shows the case when $\lambda/\alpha > 1$ and $\mu/\alpha > 1$. In Figure 5.2 are also illustrated the cases when $\lambda/\alpha = 1$ and $\mu/\alpha = 1$ and, finally, in Figure 5.3 the cases $\lambda/\alpha < 1$ and $\mu/\alpha < 1$.

In addition, if $\alpha \to \infty$, the exponentially averaged queue length process $S(t)$ will react the changes of $L(t)$ immediately, and the limiting process $\lim_{\alpha\to\infty} S(t)$ will converge to the process $L(t)$. Naturally, the limiting probability density functions of the process $S(t)$ will converge towards the pdfs of the process $L(t)$.



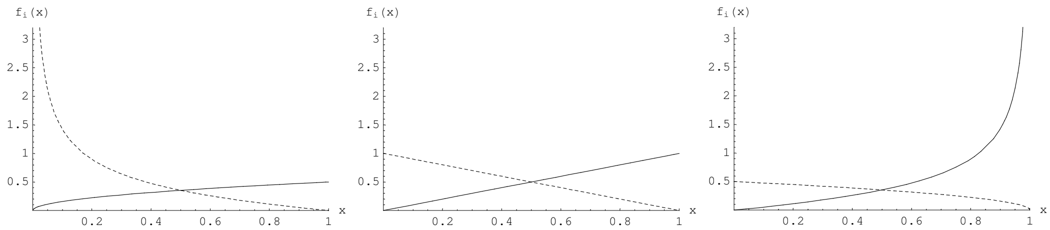
(a) $\alpha = 0.1, \lambda = 1.0$, $\mu = 2.0$ (b) $\alpha = 0.1, \lambda = 2.0$, $\mu = 2.0$ (c) $\alpha = 0.1, \lambda = 2.0$, $\mu = 1.0$

Figure 5.1: Effect of the ratio λ/μ on $f_0(x)$ (dashed) and $f_1(x)$ (solid), $\alpha = 0.1$.



(a) $\alpha = 1.0, \lambda = 1.0,\ \mu = 2.0$ (b) $\alpha = 1.0, \lambda = 2.0,\ \mu = 2.0$ (c) $\alpha = 1.0, \lambda = 2.0,\ \mu = 1.0$

Figure 5.2: Effect of the ratio λ/μ on $f_0(x)$ (dashed) and $f_1(x)$ (solid), $\alpha = 1.0$.



(a) $\alpha = 2.0, \lambda = 1.0,\ \mu = 2.0$ (b) $\alpha = 2.0, \lambda = 2.0,\ \mu = 2.0$ (c) $\alpha = 2.0, \lambda = 2.0,\ \mu = 1.0$

Figure 5.3: Effect of the ratio λ/μ on $f_0(x)$ (dashed) and $f_1(x)$ (solid), $\alpha = 2.0$.

5.3 Special solutions for M/M/1/2 system

Consider an ordinary $M/M/1/2$ queueing system with no RED mechanism involved. Similarly as described in the section 5.2, we consider an exponentially averaged queue length process driven by an $M/M/1/2$ process.

Stationary equations for the three state system

In the three state system the partial probability density functions $f_0(x)$, $f_1(x)$ and $f_2(x)$ are governed by ODE system,

$$\begin{cases} \alpha(x-0)\frac{d}{dx}f_0(x) = (\lambda-\alpha)f_0(x) - \mu f_1(x), \\ \alpha(x-1)\frac{d}{dx}f_1(x) = (\lambda+\mu-\alpha)f_1(x) - \mu f_2(x) - \lambda f_0(x), \quad 0 \le x \le 2. \\ \alpha(x-2)\frac{d}{dx}f_2(x) = (\mu-\alpha)f_2(x) - \lambda f_1(x). \end{cases} \tag{5.8}$$

Boundary conditions for the differential equations are given by

$$f_i(x) = 0, \qquad x \le 0, \quad i = 0,1,2, \tag{5.9}$$

$$F_i(x) = \int_0^x f_i(x)dx = \pi_i, \quad x \ge 2, \quad i = 0,1,2, \tag{5.10}$$

where π_i denotes the stationary probability of having i customers in an $M/M/1/2$ queue as defined in equation (5.4).

Solutions in special cases

The general solution for the three state system governed by ODE system (5.8) and boundary conditions (5.9) and (5.10) has still evaded us. However, with a special choice of the queue parameters the ODE system can be reduced into a more simple form and solved with appropriate polynomial trial functions.

Consider the symmetric case, in which

$$\lambda/\alpha = \mu/\alpha = m, \quad m = \{1, 2, 3, ...\}.$$

By differentiating the ODE system (5.8) $m-1$ times and assuming that $m+1$:th derivatives of $f_i(x), i = 0, 1, 2$ are continuous, we get an ODE system for the $f_i^{(m)}(x), i = 0, 1, 2,$

$$\begin{cases} (x-0)f_0^{(m)}(x) &= -mf_1^{(m-1)}(x), \\ (x-1)f_1^{(m)}(x) &= mf_1^{(m-1)}(x) - mf_2^{(m-1)}(x) - mf_0^{(m-1)}(x), \quad 0 \le x \le 2. \\ (x-2)f_2^{(m)}(x) &= -mf_1^{(m-1)}(x). \end{cases} \tag{5.11}$$

Differentiating the second differential equation in the ODE system (5.11) and inserting the first and the third differential equations into the result, we obtain a second order differential equation for $f_1^{(m-1)}(x)$,

$$x(x-1)(x-2)f_1^{(m+1)}(x) - (m-1)x(x-2)f_1^{(m)}(x) - 2(x-1)m^2 f_1^{(m-1)}(x) = 0. \tag{5.12}$$

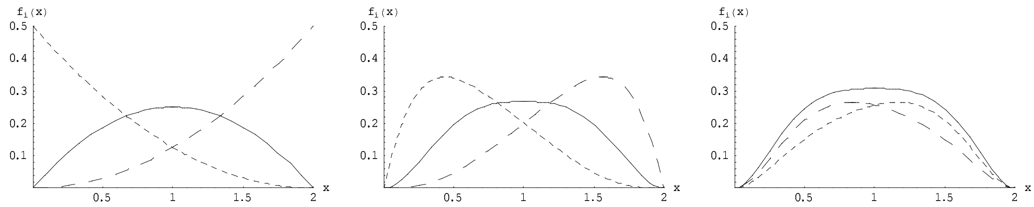
A polynomial trial function may be used to solve the DE (5.12). However, a more profound derivation of the general solution is omitted. For a fixed m, the polynomial solutions for the differential equation (5.12) can be determined with, e.g., Maple *dsolve* tool. The partial probability density functions $f_i(x), i = 0, 1, 2$ can be obtained from the solution of (5.12) with simple integration operations. The solution of $f_i(x), i = 0, 1, 2$ still has unknown coefficients that can be determined from the boundary conditions.

For odd values of m the unknown coefficients of the polynomial solutions of $f_i(x), i = 0, 1, 2$ can be determined such that the boundary conditions (5.9) and (5.10) are fulfilled. Unfortunately the solution turn out to be more complex if m is an even number and a simple polynomial function approximation satisfying the boundary conditions cannot be found.

A more profound analysis on solving the ODE system (5.8) using polynomial trial functions is presented in [17] and [18] . Based on the analysis, the solutions for the partial probability density functions are $3m - 1$ order polynomials. In addition, a solution for the case $m = 2$ is also presented. Based on the analysis, the unknown coefficients of the polynomial solution for the ODE system (5.8) must be defined separately for $0 \leq x < 1$ and $1 \leq x \leq 2$. This leads to a piecewisely defined solution that satisfies also the boundary conditions.

Illustration of the solutions

The solutions for the (5.8), i.e., the partial probability density functions $f_0(x)$, $f_1(x)$ and $f_2(x)$ in cases $m = 1$, $m = 2$ and $m = 3$ are shown in equations 5.13, 5.14, 5.15 and 5.16, respectively. The solutions partly factorizes with terms x and $(2 - x)$ and due to this, the structure of the solutions hold similarities with beta distribution. The solutions are also shown in Figure 5.4.



(a) $m = 1$ (b) $m = 2$ (c) $m = 3$

Figure 5.4: Partial probability density functions, $f_i(x), i = 0, 1, 2$ (small dashed line, solid line, large dashed lined) with $m = 1, 2, 3$

$m = 1:$

$$\begin{cases} f_0(x) &= \frac{1}{8}(2-x)^2, \\ f_1(x) &= \frac{1}{4}x(2-x), \qquad 0 \leq x \leq 2 \\ f_2(x) &= \frac{1}{8}x^2, \end{cases} \tag{5.13}$$

$m = 2:$

$$\begin{cases} f_0(x) &=& \frac{1}{15}x(3x^4 - 20x^3 + 50x^2 - 60x + 30), \\ f_1(x) &=& \frac{2}{15}x^2(-3x^3 + 15x^2 - 25x + 15), \\ f_2(x) &=& \frac{1}{15}x^3(3x^2 - 10x + 10), \end{cases} \qquad 0 \leq x \leq 1 \quad (5.14)$$

$$\begin{cases} f_0(x) &=& \frac{1}{15}(2-x)^3(3x^2 - 2x + 2), \\ f_1(x) &=& \frac{2}{15}(2-x)^2(3x^3 - 3x^2 + x + 1), \\ f_2(x) &=& \frac{1}{15}(2-x)(3x^4 - 4x^3 + 2x^2 + 4x - 2). \end{cases} \qquad 1 \leq x \leq 2 \quad (5.15)$$

$m = 3:$

$$\begin{cases} f_0(x) &=& \frac{105}{2048}x^2(2-x)^4(5x^2 - 8x + 8), \\ f_1(x) &=& \frac{105}{1024}x^3(2-x)^3(5x^2 - 10x + 8), \\ f_2(x) &=& \frac{105}{2048}x^4(2-x)^2(5x^2 - 12x + 12). \end{cases} \qquad 0 \leq x \leq 2 \quad (5.16)$$

Chapter 6

Numerical Methods to Solve the Distribution Functions

6.1 Introduction

The numerical solution of the ODE system for the stationary distribution functions (4.10) turn out to be numerically instable with traditional integration schemes. In this chapter two numerical approaches are constructed for solving the ODE system. In addition, the efficiency of the methods is briefly compared and evaluated.

6.2 Method of characteristics

The time evolution of the partial probability density functions $f_i(t,x)$ are defined with forward Kolmogorov equations,

$$\begin{aligned}\frac{\partial}{\partial t}f_i(t,x) - \alpha(x-i)\frac{\partial}{\partial x}f_i(t,x) &= \\ \lambda_{i-1}(x)f_{i-1}(t,x) - [\lambda_i(x)+\mu_i-\alpha]f_i(t,x) + \mu_{i+1}f_{i+1}(t,x), &\quad i\in\mathcal{N}, \quad (6.1)\end{aligned}$$

in which

$$\lambda_i(x) = \begin{cases} \lambda(x), & 0\le i\le K-1 \\ 0, & i=K \end{cases}, \quad \mu_i = \begin{cases} \mu, & 1\le i\le K \\ 0, & i=0 \end{cases}$$

and

$$f_i(t,x)\equiv 0, \qquad i\notin\mathcal{N}.$$

The Kolmogorov theorem guarantees that the solutions $f_i(t,x)$ of (6.1) will approach the stationary distribution $f_i(x)$ independent of the chosen initial distribution function $f_i(0,x)$. This provides an approach to numerically approximate the stationary solutions.

The idea is to construct a method to approximate the time evolution of the the distribution functions $f_i(t,x)$ over a certain time step Δt. We can start with arbitrary initial distribution $f_i(t_0,x_j)$ and using the numerical method to compute the time evolution successively over time steps Δt until the solution converges as illustrated in Figure 6.1.

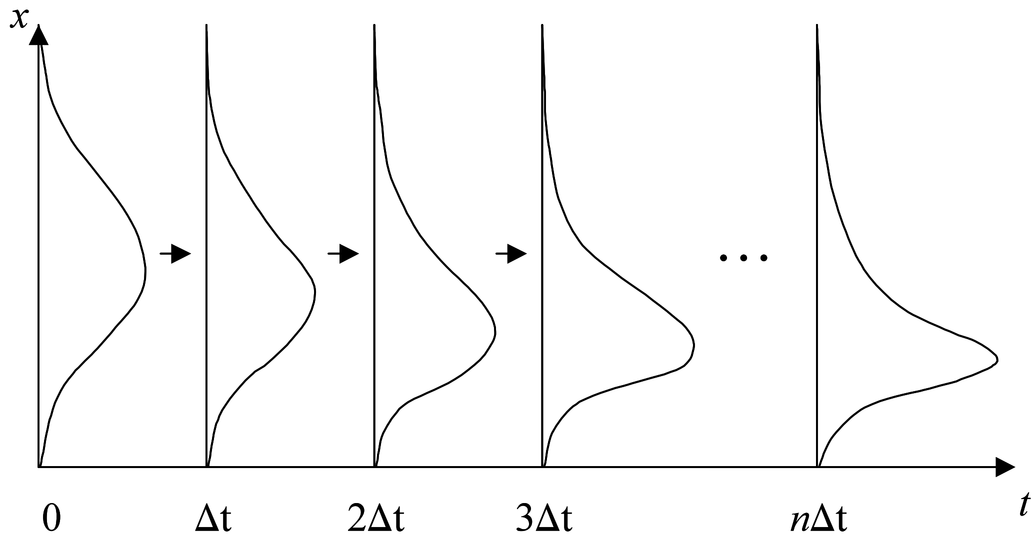


Figure 6.1: Time evolution of pdfs over successive time steps Δt.

6.2.1 Derivation of the method

The Kolmogorov equations can be transformed into a set of ordinary differential equations using the Cauchy's method of characteristics, see, e.g. [19]. For a specific i, there is a characteristic curve $x_i(t)$ following the differential equation$\frac{d}{dt}x_i(t) = -\alpha(x_i(t)-i)$,

$$x_i(t) \equiv x_i(t,x_0) = i + (x_0 - i)e^{-\alpha t}, \tag{6.2}$$

where $x_0 = x_i(0)$. Along the characteristic curves the $f_i(t, x_i(t))$ are functions of time satisfying

$$\begin{aligned}
&\frac{d}{dt} f_i(t, x_i(t)) \\
&= \frac{\partial}{\partial t} f_i(t, x_i(t)) + \frac{d}{dt} x_i(t) \frac{\partial}{\partial x} f_i(t, x_i(t)) \\
&= \frac{\partial}{\partial t} f_i(t, x_i(t)) - \alpha\,(x_i(t) - i) \frac{\partial}{\partial x} f_i(t, x_i(t)).
\end{aligned}$$

Thus the Kolmogrov equations (6.1) become ordinary differential equations,

$$\begin{aligned}
\frac{d}{dt} f_i(t, x_i(t)) &= \lambda_{i-1}(x_i(t)) f_{i-1}(t, x_i(t)) - [\lambda_i(x_i(t)) + \mu_i - \alpha] f_i(t, x_i(t)) \\
&\quad + \mu_{i+1} f_{i+1}(t, x_i(t)) \qquad (6.3) \\
&= \mathbf{Q}_i^{\mathrm{T}}(x_i(t)) \mathbf{f}(t, x_i(t)),
\end{aligned}$$

where $\mathbf{Q}_i^{\mathrm{T}}(x)$ is the ith row of the matrix $\mathbf{Q}^{\mathrm{T}}(x)$. Now, these equations can be solved by discretizing the time, $t_n = n\Delta t$, and considering the evolution over a discrete time step,

$$f_i(t_{n+1}, x) = f_i(t, x_i(-\Delta t, x)) + \int_{t_n}^{t_{n+1}} \mathbf{Q}_i^{\mathrm{T}}(x_i(t)) \mathbf{f}(t_n, x_i(t)) dt,$$

where $x_i(-\Delta t, x)$ is the position of the characteristic curve at time t_n, given that at time t_{n+1} it is at x, which by (6.2) means that $x_i(-\Delta t, x)) = i + (x - i)e^{\alpha\Delta t}$.

By the simplest integration scheme we have

$$\begin{aligned}
f_i(t_{n+1}, x) &\approx f_i(t, x_i(-\Delta t, x)) + \Delta t \cdot \mathbf{Q}_i^{\mathrm{T}}(x_i(t)) \mathbf{f}(t_n, x_i(t)) \\
&= f_i(t_n, i + (x - i)e^{\alpha\Delta t}) + \qquad (6.4) \\
&\quad \Delta t \cdot \mathbf{Q}_i^{\mathrm{T}}(i + (xi)e^{\alpha\Delta t}) \mathbf{f}(t_n, i + (x - i)e^{\alpha\Delta t}),
\end{aligned}$$

though higher order approximations could also be employed.

In practice, the state variable x has to be discretized in the interval $(0, K)$, $x_k = k\Delta x$. Then (6.5) is used to calculate the $f_i(t_{n+1}, x_k)$ at the discretization points. An interpolation can be used to calculate the values of the function for points between the discretization points. The interpolation is needed in the next time step since $(x_k - i)e^{\alpha \Delta t}$ can refer to any point between the discretization points (or to a point outside the range, where we always have $f_i(t_{n+1}, x) = 0$ for $x < 0$, and $f_i(t_{n+1}, x) = f_i(t_{n+1}, K)$ for $x > K$).

6.3 Embedded process approach

In the method of characteristics the evolution of the distribution functions $\mathbf{f}(t, x)$ was considered in continuous time. Another approach is to trace the Markov process $(L(t), S(t))$ at suitable embedded time points. One approach is to look at the process at the time instants when a transition occurs from a given state $L(t) = i$. However, more convenient is to include also the time points at which self transitions occur to the state itself. The self transitions result from the system being empty when departure transition would take place, or when arriving customer is not accepted into the system. In this case there are two parallel Poisson processes generating the transitions, the arrival process at rate λ and the service process at rate μ. Denote the time instant of the n:th transition with t_n. The time between the transitions $t_{n+1} - t_n$ is exponentially distributed as $\mathrm{Exp}(\lambda + \mu)$. Based on the PASTA property, the stationary distribution of the embedded process $(L(t_n), S(t_n))$ is then equal to stationary distribution of the continuous process $(L(t_n), S(t_n))$. In Figure 6.2 is illustrated the idea of probing the process $(L(t), S(t))$ at the embedded time points t_n .

6.3.1 Derivation of the method

For a given state x of process $S(t)$, the transition probability matrix for the process $L(t_n)$ becomes

$$\mathbf{P}(x) = \begin{pmatrix} \frac{\mu+\lambda p(x)}{\lambda+\mu} & \frac{\lambda(1-p(x))}{\lambda+\mu} & 0 & \cdots & 0 \\ \frac{\mu}{\lambda+\mu} & \frac{\lambda p(x)}{\lambda+\mu} & \frac{\lambda(1-p(x))}{\lambda+\mu} & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \frac{\mu}{\lambda+\mu} & \frac{\lambda p(x)}{\lambda+\mu} & \frac{\lambda(1-p(x))}{\lambda+\mu} \\ 0 & \cdots & 0 & \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu} \end{pmatrix}. \tag{6.5}$$

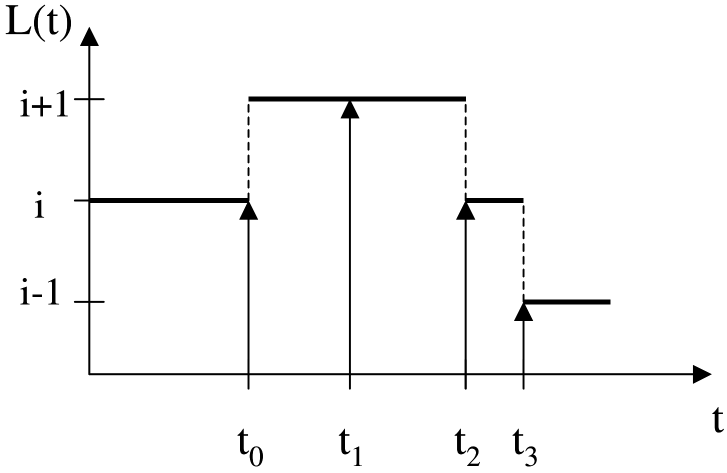


Figure 6.2: The process $(L(t), S(t))$ is monitored at time instants t_n at which transitions or self transitions occur in $L(t)$.

Denote the time instant before the transition with t_n^- and immediately after the transition with t_n^+. Let $F_{i,n}^-(x)$ and $F_{i,n}^+(x)$ be the partial cumulative distribution functions of the process (L(t),S(t)) at the time points $t = t_n^-$ and $t = t_n^+$, respectively. Denote $\mathbf{F}_n^-(x) = (F_{0,n}^-(x), F_{1,n}^-(x), ..., F_{K,n}^-(x))^{\mathrm{T}}$ and similarly for $\mathbf{F}_n^+(x)$. The evolution of the distribution functions in the transition is governed by

$$\mathbf{F}_n^+(x) = \mathbf{P}^{\mathrm{T}}(x)\mathbf{F}_n^-(x). \tag{6.6}$$

Between t_n^+ and t_{n+1}^- no transitions take place and the partial cumulative distribution functions is constant along the characteristic curve (6.2). $F_{i,n+1}^-(x)$ is obtained from $F_{i,n}^+(x)$ by

$$\begin{aligned} F_{i,n+1}^-(x) &= \int_0^\infty F_{i,n}^+(i + (x-i)e^{\alpha t})\,(\lambda+\mu)e^{-(\lambda+\mu)t}\,dt \\ &= \int_0^1 F_{i,n}^+(i + (x-i)z^{-a})\,dz, \end{aligned}$$

where the latter form, obtained by a change of variable $z = e^{-(\lambda+\mu)t}$, shows that the relation depends only on $a = \alpha/(\lambda+\mu)$ and not separately on α and $(\lambda+\mu)$. With the aid of the conditions $F_{i,n}^+(x) = 0$ for $x < 0$ and $F_{i,n}^+(x) = F_{i,n}^+(K)$ for

$x > K$, the integral can be split as follows

$$F^-_{i,n+1}(x) = \begin{cases} \displaystyle\int_{\left(\frac{i-x}{i}\right)^{1/a}}^{1} F^+_{i,n}(i-(i-x)z^{-a})dz, & x \le i, \\ \left(\frac{x-i}{K-1}\right)^{1/a} F^+_{i,n}(K) + \displaystyle\int_{\left(\frac{x-i}{K-i}\right)^{1/a}}^{1} F^+_{i,n}(i+(x-i)z^{-a})dz, & x > i. \end{cases} \tag{6.7}$$

Hence, if we define the operator

$$U_{i,a}F(x) = \begin{cases} \displaystyle\int_{\left(\frac{i-x}{i}\right)^{1/a}}^{1} F(i-(i-x)z^{-a})dz, & x \le i, \\ \left(\frac{x-i}{K-1}\right)^{1/a} F(K) + \displaystyle\int_{\left(\frac{x-i}{K-i}\right)^{1/a}}^{1} F(i+(x-i)z^{-a})dz, & x > i, \end{cases} \tag{6.8}$$

and the vector operator $\mathbf{U}$,

$$\mathbf{UF}(x) = (U_{0,a}F_0(x),\ U_{1,a}F_1(x), \ldots, U_{K,a}F_K(x))^{\mathrm{T}},$$

combination of (6.6), (6.7) and (6.8) shows that the evolution of the distribution of the embedded chain, $\mathbf{F}^-_n(x)$, can be written as

$$\mathbf{F}^-_{n+1}(x) = \mathbf{U}(\mathbf{P}^{\mathrm{T}}(x)\mathbf{F}^-_n(x)).$$

Operator $\mathbf{U}$ involves integration of the distribution functions. In practise one needs to discretize the distribution functions and use numerical integration schemes to carry out the operation.

6.4 Evaluation of the numerical methods

In this section we use the numerical approaches to approximate the joint stationary distribution of instantaneous and exponentially averaged queue length. We focus on a specific case with $M/M/1/2$ system for which the analytical solution is known. For this case, the performance of the numerical approaches is evaluated in terms of the accuracy of the solution and the computation time required to obtain the solution.

6.4.1 Comparison measures and parameterization

We have computed the cumulative distribution function

$$F(x) = \int_0^x f_0(x) + f_1(x) + f_2(x)dx$$

in the case $K = 2$, $\lambda = \mu = 1$ and $\alpha = 1/5$ with both numerical approaches. In this case we can find the solution in analytical form using the procedure described in section 5.3.

The accuracy of the numerical solution $\tilde{F}(x)$ was measured with the maximum absolute difference from the corresponding analytical solution $F(x)$,

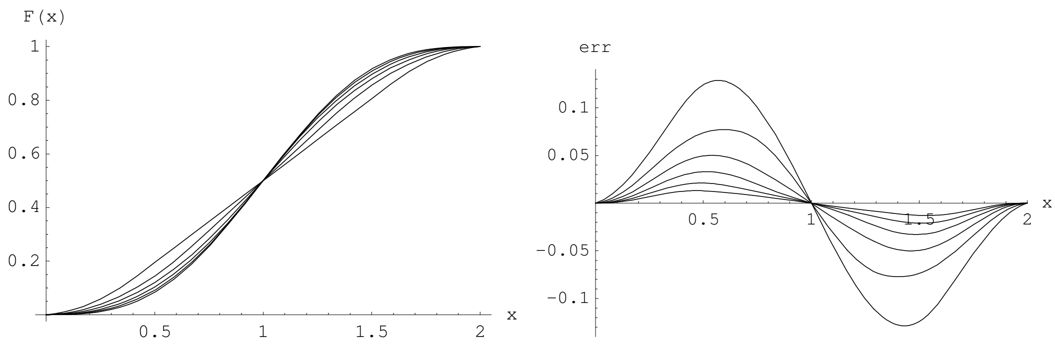
$$\varepsilon = \max_{0 \leq x \leq 2} \mid \tilde{F}(x) - F(x) \mid .$$

In order to compare the efficiency of the different numerical approaches, we measured the computation time required to numerically solve the cumulative distribution function with a desired accuracy level ε. The resolution of discretization and the other parameters of the numerical approaches affect both the accuracy and the computational efficiency of the method. The method of characteristics and the embedded process approaches are based on solving the time dependent equations on certain interval $[0, T]$ and using the solution at time instant T as an approximation for the stationary distribution. In addition to setting the length of time frame T, one has to choose the initial distribution function and also to set the resolution of the state space discretization in both of these methods. In the embedded process approach the time evolution of the distribution functions is inherently computed over the time step between successive embedded time points. However, in the case of the method of characteristics one needs to define a time step Δt over which the evolution of the distribution functions are computed.

The differences between the numerical approaches complicate the comparison of the methods. The discretization of both the state and the time space are chosen as coarse as possible and the time frame T as small as possible such that the desired accuracy level can still be achieved. When the parameters of the numerical methods are chosen in this manner the computation times correspond to the minimum time required to solve the stationary distributions. However, it must be emphasized that the actual implementation of the methods may still effect the computation time considerably.

6.4.2 Results

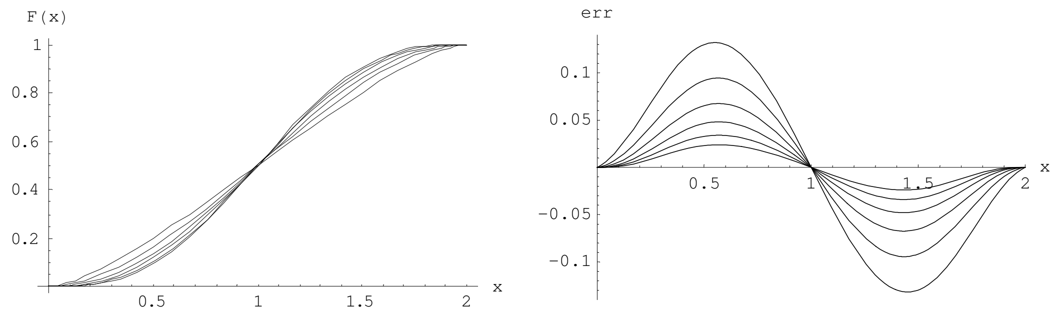
The effect of the length of the time frame T on the numerical solutions is illustrated in Figures 6.3 and 6.4. Figure 6.3 illustrates the evolution of the numerical solution $\tilde{F}(x)$ and the error curve $\tilde{F}(x) - F(x)$ for the method of characteristics with time frames $T = \{1, 2, 3, 4, 5, 6\}$ and Figure 6.4 respectively for the embedded process approach. The state space is divided into 40 intervals each of length $\Delta x = 0.05$ in both approaches. The time step for the method of characteristics was set to $\Delta t = 0.05$. The initial distribution function was taken to be uniform both in the method of characteristics and in the embedded process approach. Based on the evaluation, the error decreases roughly exponentially as the time frame T increases.



(a) Evolution of $\tilde{F}(x)$ (b) Evolution of the error $\tilde{F}(x) - F(x)$

Figure 6.3: Method of characteristics: numerical solution with $T = \{1, 2, 3, 4, 5, 6\}$.

We have chosen the maximum allowed error level $\varepsilon = 10^{-3}$ in the further analysis. The numerical approaches were implemented with Mathematica 4.1 and the test runs were carried out on a PC equipped with AMD Athlon 1400 MHz processor and 512 MB of memory. In Table 6.1, we show the computation time required to solve the problem with desired accuracy level ε. Also the required resolution of the discretization and the length of a time frame is



(a) Evolution of $\tilde{F}(x)$ (b) Evolution of the error $\tilde{F}(x) - F(x)$

Figure 6.4: Embedded process approach: numerical solution with $T = \{1, 2, 3, 4, 5, 6\}$.

shown in the table. Based on the results, the method of characteristics is found to be roughly ten times faster than the embedded process approach.

Method	**Computation Time (s)**	Δx	T	Δt
Method of characteristics	7	0.05	16	0.02
Embedded process approach	103	0.05	15	-

Table 6.1: Comparison of different numerical approaches.

The implementation of the numerical methods can effect the computation times considerably. Because of this, the results give only a hint about the magnitude of computation time required by the approaches. In addition, the parameters of the queueing system itself affect the computation times. With larger buffer size K the computation time increases with both of the methods. Furthermore, small values of the parameter α may also increase the computation time.

Chapter 7

Numerical Example

7.1 Introduction

In this chapter we solve numerically the stationary probability distribution functions for $M/M/1/20$ queue with RED packet dropping mechanism for one specific parameter combination. The results are compared with the data from a simulation model in order to verify the numerical solution.

7.2 Stationary properties of a simple queue with RED mechanism

Consider an $M/M/1/K$ buffer with RED mechanism. The buffer has 20 queueing positions, i.e., $K = 20$. The packets arrive to the buffer according to the Poisson process with rate $\lambda = 1.0$. The packet service rate $\mu = 1.0$, i.e., the traffic load into the buffer equals with the output link capacity. The parameters for the RED congestion control mechanism are determined such that the averaging constant $\alpha = 0.1$. The packet dropping probability function in terms of the exponentially averaged queue length x is defined

$$p(x) = \begin{cases} 0, & 0 \leq x < th_{min}, \\ p_{max} \frac{x - th_{min}}{th_{min} - th_{max}}, & th_{min} \leq x < th_{max}, \\ 1, & th_{max} \leq x \leq 20, \end{cases}$$

in which $th_{min} = 10$, $th_{max} = 16$ and $p_{max} = 0.5$. The packet dropping probability function $p(x)$ is shown in Figure 7.1.

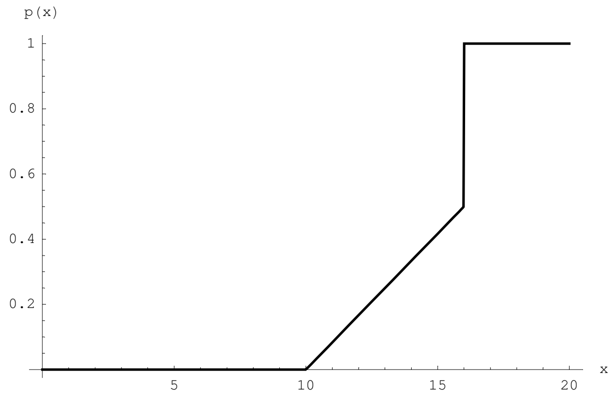


Figure 7.1: Packet dropping probability function $p(x)$ used in the example.

7.2.1 Numerical results

The stationary properties of the above buffer system were determined using the buffer model derived in Chapter 4, i.e. we consider a Markov process $(L(t), S(t))$ in which $L(t)$ is the instantaneous queue length and $S(t)$ is the exponentially averaged queue length.

The stationary partial probability density functions $f_i(x),\ i = 0, \ldots, 20$, were solved numerically using the method of characteristics. The discretization step was set to $\Delta x = 0.05$, the time step to $\Delta t = 0.05$ and the time frame parameter to $T = 150$.

The numerical results of $f_i(x),\ i = 0, \ldots, 20$, are illustrated in Figure 7.2. The effect of the step in the dropping probability function $p(x)$ at point $x = 16$ is clearly visible in the probability distribution functions.

7.3 Comparison with simulation results

A simulation model was constructed to verify the numerical solution of the analytical model. In the simulation, the incoming packets are generated from Poisson source with intensity $\lambda = 1.0$ and the packet sizes are exponentially distributed with mean $\mu = 1.0$. Thus the simulation model resembles the analytical model exactly.

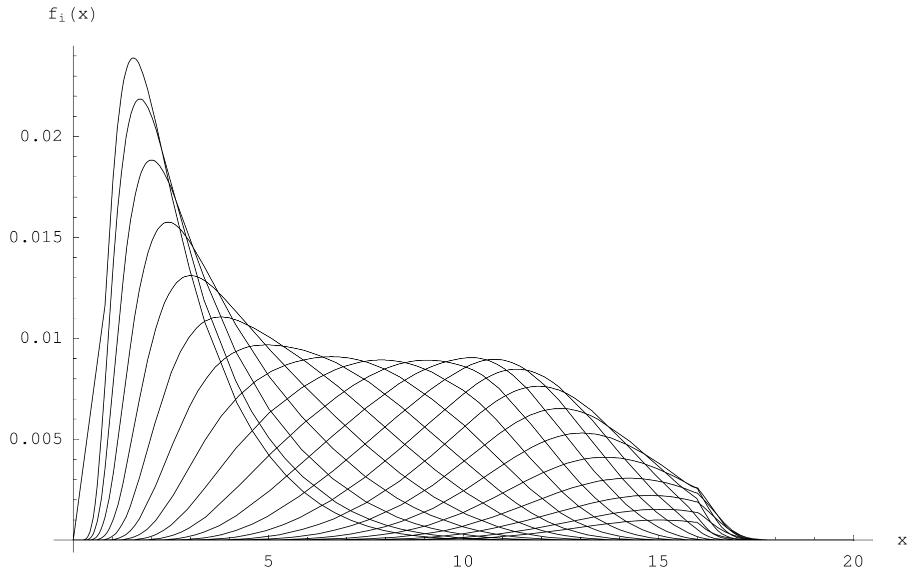


Figure 7.2: Stationary partial distribution functions $f_i(x), i = 0, \ldots, 20$ (from left to right).

7.3.1 Computing histogram from the simulation data

The values of the instantaneous and exponentially averaged queue lengths at the time of the packet arrival were collected during the simulation run. Due to the PASTA property the simulation data resembles samples from the stationary process $\lim_{t\to\infty}(L(t), S(t))$.

A histogram of the values of exponentially averaged queue length was computed from the simulation data. Let S_k denote the value of the exponentially averaged queue length process $S(t)$ at k:th packet arrival (k_{max} arrivals are considered) in the simulation data. We can define the discretized histogram function $h(x_n)$ for each $n = 0, \ldots, N-1$,

$$h(x_n) = \#\text{ samples, for which } S_k \in [x_n, x_{n+1}), \quad k = 0, \ldots, k_{max},$$

in which N denotes the number of discretization levels of the histogram function and $x_n = n\frac{K}{N} = n\Delta x$.

By normalizing the histogram function with a scaling parameter c, such that

$$c\int_0^K h(x)dx = c\sum_{n=0}^{N-1} h(x_n)\Delta x = 1,$$

the $c \cdot h(x_n)$ approximates the probability $\lim_{t\to\infty} P\{S(t) \in [x_n, x_{n+1})\} \approx f(x_n) = \sum_{i=0}^{K} f_i(x)$. Thus, the normalized histogram values can be used as reference data to verify the numerical solution of the probability density function $f(x)$.

7.3.2 Comparison

Five independent simulation runs were carried out and the histogram for the exponentially averaged queue length values were computed for each run. Each run consisted of one million arrivals, i.e. $k_{max} = 1000000$. The histogram was discretized with parameter $N = 200$. An average histogram was computed from this data and then normalized. In addition, assuming that the simulated histogram values $h(x_n)$ for each simulation run are drawn from a normal distribution with certain mean and variance, a 95% confidence interval for the average histogram values were computed.

In Figure 7.3 the simulated histogram with the confidence intervals and the numerical solution are shown. The numerical solution follows the simulated histogram values very accurately.

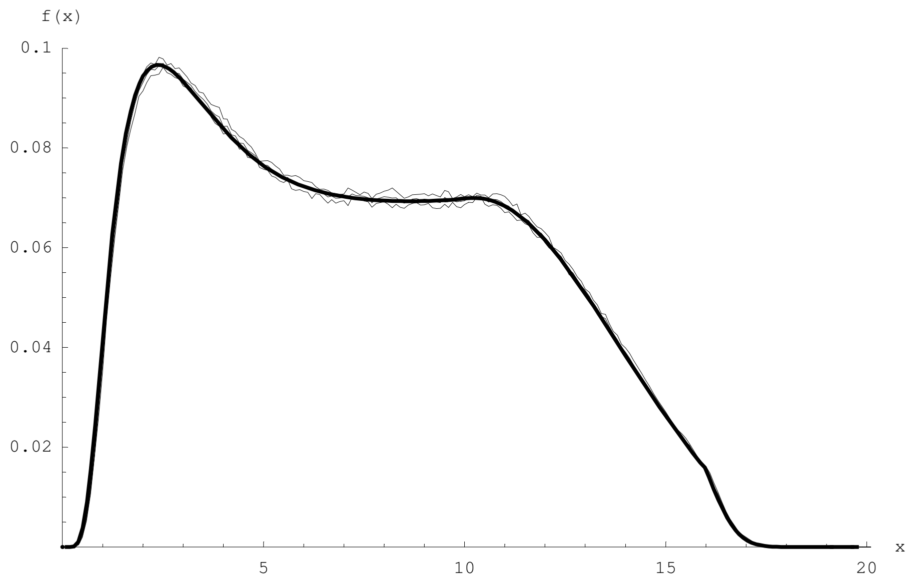


Figure 7.3: Simulated histogram values with 95% confidence intervals (thin lines) and the numerical solutions for $f(x)$ (bold line).

Chapter 8

Model Extensions

In this chapter two extensions for the basic model of the joint system of instantaneous and exponentially averaged queue derived in section 4.2 are briefly introduced. First in section 8.1 we show how a more complex traffic model, namely a Markov Modulated Poisson Process, can be used to represent a bursty traffic source. Secondly, in section 8.2 we illustrate how the basic model can be extended to a buffer system with several coupled queues, such as an Assured Forwarding PHB buffer.

8.1 Markov modulated traffic sources

In the basic model of the joint system of instantaneous and exponentially averaged queue it is assumed that the incoming traffic is generated according to a Poisson process. However, a more complex Markov Modulated Poisson Process (MMPP) traffic model (see, e.g., [20]) can be attached to the basic model with rather simple modifications.

In the MMPP the packet arrival rate is regulated by an arbitrary Markov process $M(t)$ with state space $\mathcal{M} = \{m|m = \{0, 1, \ldots, M\}\}$. If $M(t) = m$ the packets are generated according to a Poisson process with a certain arrival rate λ_m. The time between the arrival rate changes is exponentially distributed, i.e. the arrival rate is Markov modulated. A buffer system driven by MMPP traffic source is illustrated in Figure 8.1.

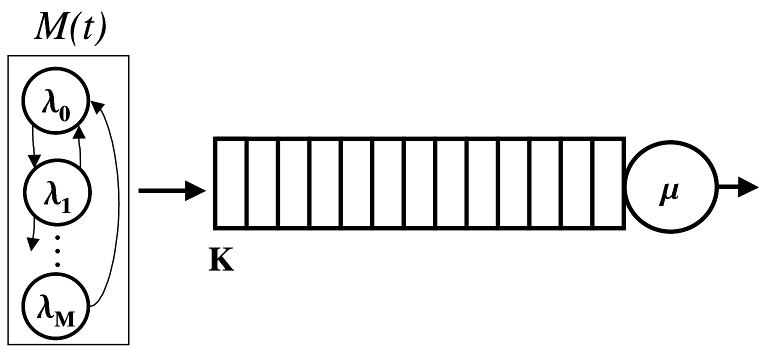


Figure 8.1: Illustration of buffer system with MMPP traffic source.

8.1.1 An extended model with MMPP traffic sources

In order to include the behavior of the MMPP traffic source into the model, we need to consider the combined three dimensional process $(M(t), L(t), S(t))$ with the state space

$$\mathcal{S}_{MLS} = \{(m, i, x) | m \in \{0, \ldots, M\},\ i \in \{0, \ldots, K\},\ x \in [0, K]\},$$

in which $L(t)$ and $S(t)$ are the instantaneous and the exponentially averaged queue length process, respectively, as defined in section 4.2. The process $(M(t), L(t), S(t))$ constitutes a Markov process and the modeling approach presented in the previous chapter can be adapted.

Kolmogrov equations

Denote the partial cumulative distribution function of process $(M(t), L(t), S(t))$ with

$$F_{m,i}(t, x) = P\{M(t) = m, L(t) = i, S(t) \leq x\}, \quad m \in \mathcal{M},\ i \in \mathcal{N},$$

and the partial probability density function with

$$f_{m,j}(t, x) = \frac{\partial}{\partial x} F_{m,i}(t, x), \quad m \in \mathcal{M},\ i \in \mathcal{N}.$$

The Kolmogorov equations for the process $(M(t), L(t), S(t))$ are similar to the equations of the basic model,

$$\frac{\partial}{\partial t}\mathbf{f}(t, x) + \mathbf{D}'(x)\frac{\partial}{\partial x}\mathbf{f}(t, x) = \mathbf{Q}'^T(x)\mathbf{f}(t, x), \tag{8.1}$$

in which

$$\mathbf{f}(t,x) = (f_{0,0}(t,x),\ldots,f_{0,K}(t,x),f_{1,0}(t,x),\ldots,f_{1,K}(t,x),,\ldots,f_{M,K}(t,x))^T,$$

$$\mathbf{D}'(x) = \begin{pmatrix} \mathbf{D}(x) & & & \mathbf{0} \\ & \mathbf{D}(x) & & \\ & & \ddots & \\ \mathbf{0} & & & \mathbf{D}(x) \end{pmatrix},$$

where $\mathbf{D}(x)$ is rate matrix as defined in section 4.2 and $\mathbf{Q}'(x)$ is the transition rate matrix of the two-dimensional process $(M(t), L(t))$. The exact form of $\mathbf{Q}'(x)$ depends on the structure of the process $M(t)$.

Stationary equations

In the stationary case we get

$$\mathbf{D}'(x)\frac{d}{dx}\mathbf{f}(x) = \mathbf{Q'}^T(x)\mathbf{f}(x), \tag{8.2}$$

in which $\mathbf{f}(x) = \lim_{t\to\infty}\mathbf{f}(t,x)$.

8.2 A buffer system with coupled queues

In the following the basic model is extended to a system with two coupled queues. By coupled queues we mean a system in which the queue share the available output link bandwidth according to predefined weights. If one queue is empty the other queue can utilize all the available bandwidth of the output link. This is also known as the Generalized Processor Sharing (GPS) scheduling principle [21]. The approach presented here can be followed to model even larger buffer systems, e.g., the Assured Forwarding PHB scheme that consists of four coupled queues.

8.2.1 An extended model for two coupled queues

A buffer system with two coupled queues is illustrated in Figure 8.2. Mark queues with $b \in \mathcal{B} = \{1,2\}$. The queue lengths are K_1 and K_2. It is assumed that the packets arriving to the buffers are generated by two independent Poisson processes with intensities λ_1 and λ_2. The service intensity μ is divided between the buffers according to the weight parameters w_1 and w_2, such that $w_1 + w_2 = 1$, $w_1, w_2 \geq 0$. To keep the problem tractable it is assumed that the output link can serve the packets in both buffers at the same time. Thus, the service intensity for buffer b is defined $\mu_b = w_b\mu$.

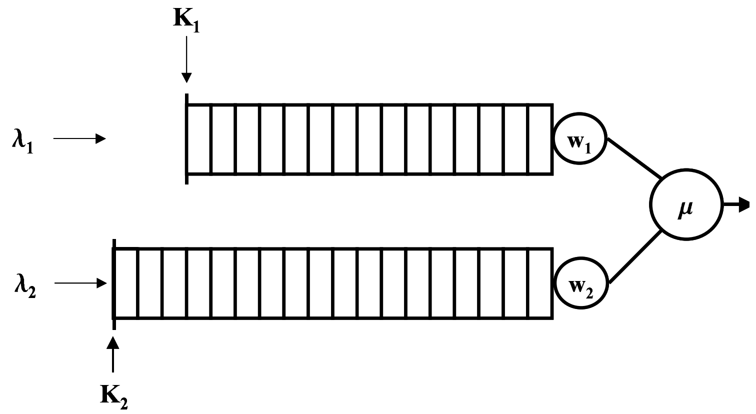


Figure 8.2: Illustration of the buffer system with two coupled queues.

Let $L_1(t)$ denote the instantaneous queue length process of the queue one with a state space $\mathcal{N}_1 = \{i | i = \{0, \ldots, K_1\}\}$. $S_1(t)$ denotes the exponentially averaged queue length process of queue 1. Correspondingly we have processes $L_2(t)$ with state space $\mathcal{N}_2 = \{j | j = \{0, \ldots, K_2\}\}$ and $S_2(t)$ for the second queue. The processes $S_1(t)$ and $S_2(t)$ follow differential equation (4.2). It is assumed that the averaging constant α is the same for both averaging processes.

The coupled two queue system can be described with a four dimensional process $(L_1(t), L_2(t), S_1(t), S_2(t))$ with state space

$$\mathcal{S}_{LLSS} = \{(i, j, x, y) | i \in \{0, \ldots, K_1\}, j \in \{0, \ldots, K_2\}, x \in [0, K_1], y \in [0, K_2]\}.$$

Also in this case the state transition probabilities of the above process depends only on the current state of the process. Thus, the process $(L_1(t), L_2(t), S_1(t), S_2(t))$ constitutes a Markov process.

The joint state transition diagram for the instantaneous queue length processes $L_1(t)$ and $L_2(t)$ is shown in Figure 8.3. The state transition rates depend on the current states of processes $S_1(t) = x$ and $S_2(t) = y$ such that $\lambda_1(x) = \lambda_1(1 - p_1(x))$ and $\lambda_2(y) = \lambda_2(1 - p_2(y))$. The $p_1(x)$ and $p_2(y)$ are the dropping probabilities associated with the averaged queue lengths of the queue one and the queue two, respectively. The state transition diagram for the instantaneous queue length processes $(L_1(t), L_2(T))$ is illustrated in the Figure 8.3.

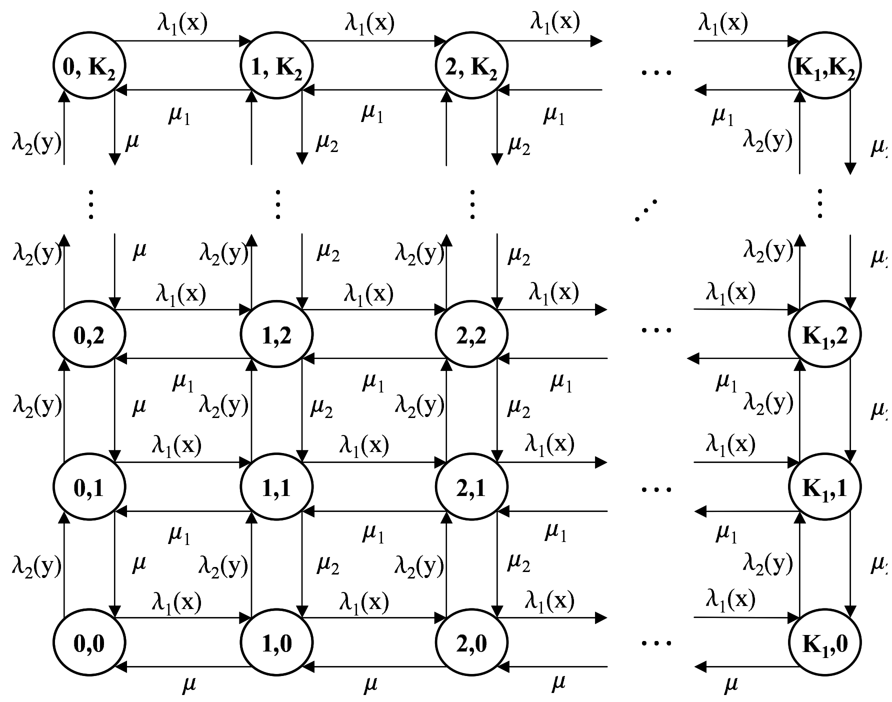


Figure 8.3: State transition diagram of $(L_1(t), L_2(t))$.

Kolmogorov equations

Denote the two dimensional partial cumulative distribution function of the process $(L_1(t), L_2(t), S_1(t), S_2(t))$ with

$$F_{i,j}(t, x, y) = P\{L_1(t) = i, L_2(t) = j, S_1(t) < x, S_2(t) < y\},\ i \in \mathcal{N}_1,\ j \in \mathcal{N}_2,$$

and the partial probability density function with

$$f_{i,j}(t, x, y) = \frac{\partial^2}{\partial x \partial y} F_{i,j}(t, x, y), \quad i \in \mathcal{N}_1,\ j \in \mathcal{N}_2$$

Using the similar argumentation as in chapter 4.2 we can derive the forward Kolmogorov equations for the partial cumulative distribution functions

$F_{i,j}(t,x,y)$,

$$
\begin{aligned}
\frac{\partial}{\partial t}F_{i,j}(t,x,y) - \alpha(x-i)\frac{\partial}{\partial x}F_{i,j}(t,x,y) - \alpha(y-j)\frac{\partial}{\partial y}F_{i,j}(t,x,y) &= \\
-\int_0^y\int_0^x (\lambda_1(i,s)+\lambda_2(j,v)+\mu_1(i)+\mu_2(j))f_{i,j}(t,s,v)ds\ dv & \\
+\int_0^y\int_0^x \lambda_1(i-1,s)f_{i-1,j}(t,s,v)+\mu_1(i+1)f_{i+1,j}(t,s,v)ds\ dv & \\
+\int_0^y\int_0^x \lambda_2(j-1,v)f_{i,j-1}(t,s,v)+\mu_2(j+1)f_{i,j+1}(t,s,v)ds\ dv, & \\
i\in\mathcal{N}_1,\ j\in\mathcal{N}_2. &
\end{aligned} \tag{8.3}
$$

in which

$$
\lambda_1(i,x) = \begin{cases} \lambda_1(x), & i\in\mathcal{N}_1/\{K_1\}, \\ 0, & i=K_1, \end{cases} \qquad \mu_1(i) = \begin{cases} \mu_1, & i\in\mathcal{N}_1/\{0\}, \\ 0, & i=0, \end{cases}
$$

and

$$
\lambda_2(j,y) = \begin{cases} \lambda_2(y), & j\in\mathcal{N}_2/\{K_2\}, \\ 0, & j=K_2, \end{cases} \qquad \mu_2(j) = \begin{cases} \mu_2, & j\in\mathcal{N}_2/\{0\}, \\ 0, & j=0, \end{cases}
$$

and

$$
F_{i,j}(t,x) \equiv 0, \quad i\notin\mathcal{N}_1,\ j\notin\mathcal{N}_2.
$$

Again, assuming the continuity of the second derivatives of $F_{i,j}(t,x,y)$, the Kolmogorov equations can be written as a set of partial differential equations for the partial probability density functions $f_{i,j}(t,x,y)$,

$$
\begin{aligned}
\frac{\partial}{\partial t}f_{i,j}(t,x,y) - \alpha(x-i)\frac{\partial}{\partial x}f_{i,j}(t,x,y) - \alpha(y-j)\frac{\partial}{\partial y}f_{i,j}(t,x,y) &= \\
-(\lambda_1(i,x)+\lambda_2(j,y)+\mu_1(i)+\mu_2(j)-2\alpha)f_{i,j}(t,x,y) & \\
+\lambda_1(i-1,x)f_{i-1,j}(t,x,y)+\mu_1(i+1)f_{i+1,j}(t,x,y) & \\
+\lambda_2(j-1,y)f_{i,j-1}(t,x,y)+\mu_2(j+1)f_{i,j+1}(t,x,y), & \\
i\in\mathcal{N}_1,\ j\in\mathcal{N}_2, &
\end{aligned} \tag{8.4}
$$

and

$$
f_{i,j}(t,x,y) \equiv 0, \quad i\notin\mathcal{N}_1,\ j\notin\mathcal{N}_2.
$$

Stationary equations

In the stationary case we have $\frac{\partial}{\partial t} f_{i,j}(t,x) \equiv 0, i \in \mathcal{N}_1,\ j \in \mathcal{N}_2$. Applying this onto equation (8.5) and defining

$$f_{i,j}(x) = \lim_{t\to\infty} f_{i,j}(t,x), \quad i \in \mathcal{N}_1,\ j \in \mathcal{N}_2,$$

we get a set of partial differential equations governing the stationary distribution of the process

$$\begin{aligned}
\alpha(x-i)\frac{\partial}{\partial x} f_{i,j}(x,y) + \alpha(y-j)\frac{\partial}{\partial y} f_{i,j}(x,y) \quad &= \\
+(\lambda_1(i,x) + \lambda_2(j,y) + \mu_1(i) + \mu_2(j) - 2\alpha) f_{i,j}(x,y) & \\
-\lambda_1(i-1,x) f_{i-1,j}(x,y) - \mu_1(i+1) f_{i+1,j}(x,y) & \qquad (8.5) \\
-\lambda_2(j-1,y) f_{i,j-1}(x,y) - \mu_2(j+1) f_{i,j+1}(x,y), & \\
i \in \mathcal{N}_1,\ j \in \mathcal{N}_2. &
\end{aligned}$$

and

$$f_{i,j}(x,y) \equiv 0, \quad i \notin \mathcal{N}_1,\ j \notin \mathcal{N}_2.$$

The above reasoning can be followed also for buffer systems with more than two coupled queues. The complexity of the model naturally increases, however, the structure of the mode remains the same.

8.3 Remarks about solving the extended models

The extended models are difficult to solve analytically or numerically with traditional integration schemes. However, despite the extensions, the model structure remains rather similar such that the numerical approaches for solving the basic model can be utilized with only slight modifications.

8.3.1 Model with Markov modulated traffic sources

The structure of the model with MMPP traffic sources remains the same as the basic model. However, instead of $K+1$ ordinary differential equations we now have $(K+1)(M+1)$ ordinary differential equations. In the stationary case we can solve the model numerically by applying directly the method of characteristic or the embedded process approach.

8.3.2 Model with coupled queues

The buffer model for two coupled queues is slightly more complex as the probability distribution functions are governed by a set of partial differential equations (8.6). However, the idea of the method of characteristics introduced in section 6.2 can be applied now in two dimensional space. The numerical approach extends also for a buffer systems with more than two coupled queues.

Chapter 9

Conclusions

9.1 Discussion

In this thesis, a modeling methodology for a DiffServ network node with Assured Forwarding PHB was introduced. The essential feature of the AF buffer is the packet dropping mechanism based on exponentially averaged queue length, such as the RED scheme. The developed model captures the dynamics involved in this mechanism.

A Markovian fluid queue model for the joint dynamics of the instantaneous and related exponentially averaged buffer length was constructed. The time evolution of the joint distribution functions of the instantaneous and averaged queue length was described with Kolmogorov equations. In the stationary case the Kolmogrov equations were transformed into a system of ordinary differential equations for the joint distribution functions.

An analytical solution for the distribution functions was found only in a few special cases in which there is no packet dropping mechanism involved. The numerical solution of the ODE system turned out to be unstable when traditional integration schemes and other numerical methods were considered. Two different numerical approaches were presented to approximate the stationary solutions. The idea in the numerical methods is based on the Kolmogrov theorem that guarantees that any initial distribution function approaches the stationary distribution as time evolves. Thus, approximating the evolution of the distribution functions over a certain small time interval and repeating the approximation successively, we eventually get an accurate approximation for the stationary distribution functions. In the first numerical method, the method of characteristics, the time evolution was considered in continuous time. In the second method, the embedded process approach, the evolution of the distribution functions was computed over embedded time points.

The stationary distributions were obtained with numerical methods in special cases for which analytical results were available. Comparison between the numerical methods was carried out. Based on the results, both the approaches can be used to obtain accurate approximations for the stationary distributions. In addition, the stationary distribution functions for a moderate size buffer with RED mechanism were solved with the continuous time method. The numerical solution was verified with data gathered from a simulation model.

Two extensions to the basic model were described. A more complex traffic model based on Markov Modulated Poisson Process sources was incorporated into the model. Such a traffic model could describe bursty traffic sources. More importantly, it was shown how the basic, single queue buffer model can be easily extended to a buffer system with two or more coupled queues. Such a model provides a tool to analyze AF buffer dynamics in a more versatile manner. Though the extended models are considerably more complex than the basic model, the basic structure of the model remains the same. Thus the numerical approaches require only slight modifications to be adapted into the extended models.

9.2 Further work

In this thesis a modeling methodology for the AF Buffer was introduced. The model can be readily used in analyzing a single queue of an AF buffer as illustrated in the numerical example. However, with minor modifications the model can be extended to cover the full AF buffer with four queues or more complex traffic sources. The model becomes very complex in these cases and solving such models numerically can be very time consuming. Thus, more research needs to be carried out to improve the efficiency and to speed up the numerical approaches described in the thesis.

Another direction in further research efforts is to find the analytical solutions for the distribution functions. Currently analytical solutions for the partial probability density functions can be found only in a few special cases. These solutions may give hint about the structure of the general solution. However, more research needs to be carried out in order to fully understand the behavior of the ODE system governing the probability density functions.

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Appendix A

Random Early Detection (RED)

Introduction

Random Early Detection mechanism for congestion avoidance in packet-switched networks was first introduced in [7] by Floyd and Jacobson. The purpose of the RED mechanism is to accompany TCP congestion control mechanism in order to avoid global synchronization of the TCP connections. The RED routers detects incipient congestion by keeping track of the averaged queue size. The RED router then implicitly notifies the traffic sources of congestion either by dropping packets or by marking the packets. When the averaged queue length exceeds a preset threshold, the router drops or marks each arriving packet with certain probability, where the exact probability is a function of the averaged queue size.

RED algorithm

The RED algorithm consists of two main elements. The first element is the computation of the averaged queue length and second one is the packet marking or dropping algorithm based on the averaged queue length.

Computation of averaged queue length

The RED router computes the average queue length avg using a low-pass filter with an exponential weighted moving average. After each packet arrival the value of the avg is updated according to

$$avg = (1 - w_q)avg + w_q q,$$

in which q is the instantaneous queue length and w_q is the time constant, or an averaging constant of the low-pass filter. If the queue is empty when packet arrives, the algorithm take into account the the period when the queue is empty by estimating the number of packets m that could have been transmitted by the router during the idle period of length t_{idle}. After the idle period, the router computes the average queue size as if m packets had arrived to an empty queue,

$$\begin{aligned} m &= f(t_{idle}), \\ avg &= (1 - w_q)^m avg, \end{aligned}$$

in which $f(t)$ is a linear function of time t.

Packet marking mechanism

The packet marking mechanism is based on the averaged queue length avg. The average queue length is compared to two thresholds, minimum and maximum threshold th_{min} and th_{max}. When the average queue length is less than the minimum threshold, no packets are marked. When the maximum threshold is exceeded, every arriving packet is marked. When the average queue length is between the the minimum and maximum thresholds, each arriving packet is marked with the probability p_a, where p_a is a function of the average queue length avg. As avg varies from th_{min} to th_{max}, the packet marking probability increases linearly from 0 to a certain p_{max}. However, there is a counter mechanism involved that increases slowly the packet marking probability from the last marked packet,

$$\begin{aligned} p_b &= p_{max}(avg - th_{min}/(th_{max} - th_{min})), \\ p_a &= p_b/(1 - count \cdot p_n), \end{aligned}$$

in which $count$ is increased by 1 with each arriving packet if the packet is not marked and, in case the packet is marked, the $count$ is set to 0. The counter mechanism ensures that the router does not wait too long before marking a packet.