Helsinki University of Technology
Signal Processing Laboratory

S-38.411 Signal Processing in Telecommunications I

Spring 2000

Lecture 3: Transmit and receive filters for bandlimited AWGN channels

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Timetable

- **L1** Introduction; models for channels and communication systems
- **L2** Channel capacity
- **L3 Transmit and receive filters for bandlimited AWGN channels**
- **L4** Optimal linear equalizers for linear channels 1
- **L5** Optimal linear equalizers for linear channels 2
- **L6** Adaptive equalizers 1
- **L7** Adaptive equalizers 2
- **L8** Nonlinear receivers 1: DFE equalizers
- **L9** Nonlinear receivers 2: Viterbi algorithm
- **L10 GL1**: DSP for Fixed Networks / *Matti Lehtimäki, Nokia Networks*
- **L11 GL2**: DSP for Digital Subscriber Lines / *Janne Väänänen, Tellabs*
- **L12 GL3**: DSP for CDMA Mobile Systems / *Kari Kalliojärvi, NRC*
- **L13** Course review, questions, feedback
- **E** 24.5. (Wed) 9-12 S4 **Exam**

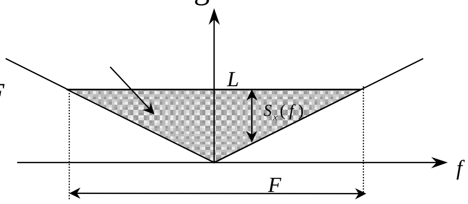
Contents of Lecture 3

I. Ideal sinc solution

Ideal sinc solution

- In the previous lecture, we discussed the basic limits for data transmission in practical channels
- In order to achieve the capacity, the power spectrum $S_x(f)$ of the transmitted signal must be chosen right:

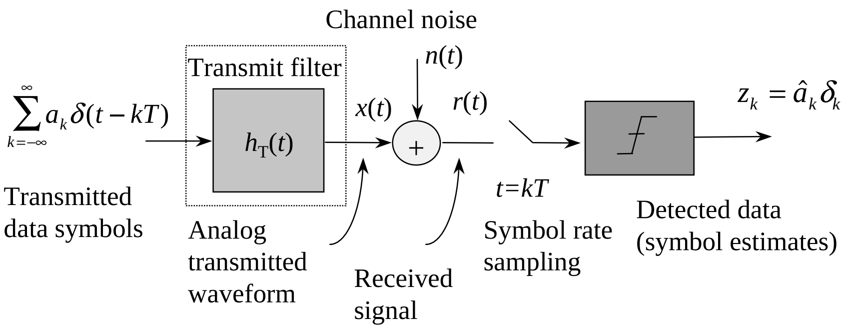
$$S_{x,opt}(f) = \begin{cases} L - \dfrac{S_n(f)}{|C(f)|^2}, & f \in F \\ 0, & \text{elsewhere} \end{cases}$$



- How can we design signals with desired power spectrum?
- How should the receiver (pre)process the received signal for detection (estimation) of transmitted data symbols?

Ideal sinc solution...

- System model:



Ideal sinc solution...

- Structure of transmitted signal with *linear modulation* methods (PAM, QAM, MPSK, etc.):

$$x(t) = h_{\mathrm{T}}(t) * \sum_{k=-\infty}^{\infty} a_k \delta(t - kT) = \sum_{k=-\infty}^{\infty} a_k h_{\mathrm{T}}(t - kT)$$

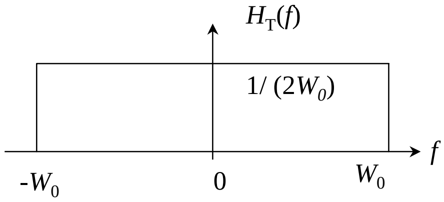
a_k = data symbols to be transmitted
$h_{\mathrm{T}}(t)$ = transmitted continuous-time waveform
$\delta(t)$ = Dirac delta function

- a_k are assumed to be uncorrelated (white spectrum) $\rightarrow$ Tx power spectrum determined by $h_{\mathrm{T}}(t)$ only!

Ideal sinc solution...

- Assume AWGN channel
- According to the water pouring theorem, the optimal Tx PSD is *constant* in the signal bandwidth W_0
- Choose rectangular signal spectrum:

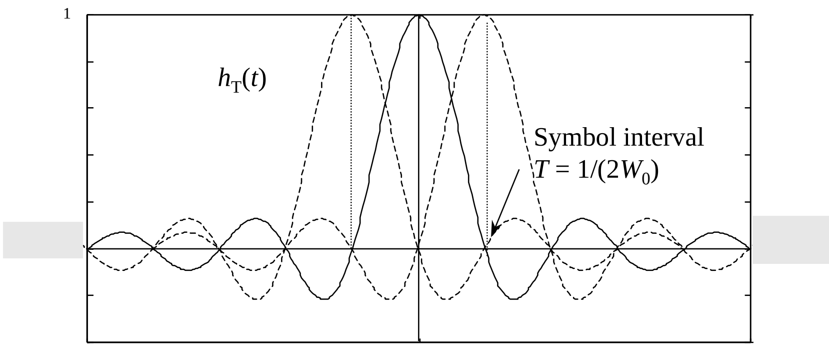
$$H_{\mathrm{T}}(f) = \frac{1}{2W_0} \mathrm{rect}\left(\frac{f}{2W_0}\right) = \begin{cases} \frac{1}{2W_0}, & |f| < W_0 \\ 0, & \text{elsewhere} \end{cases}$$



Ideal sinc solution...

- Ideal time-domain waveform via IFT:

$$h_T(t) = \int_{-\infty}^{\infty} H_T(f) e^{j2\pi ft} df = \frac{1}{2W_0} \int_{-W_0}^{W_0} e^{j2\pi ft} df = \mathrm{sinc}(2W_0 t)$$



Ideal sinc solution...

- $h_T(t) = \mathrm{sinc}(2W_0 t) = \mathrm{sinc}(t/T)$ ($\mathrm{sinc}(x)=\sin(\pi x)/(\pi x)$)
- $h_T(t) = 0$, $t = kT$ (except $h_T(0) = 1$)

  → no *intersymbol interference* (*ISI*) with any linear modulation method (PAM, QAM, etc.)

  (symbol rate $R_S = 1/T$, ideal sampling at the receiver)

Problems:

- strictly rectangular spectrum hard to implement
  - slowly decaying, infinitely long, noncausal waveform
  - sensitivity to *timing errors*
  - impulse response must be *truncated* in digital implementation → truncation errors (Gibbs phenomenon)

II. Nyquist criterion

Nyquist criterion

- Use of the flat power spectrum for the (only) design criterion leads into sinc function with:
  - zero ISI (good!)
  - implementation problems due to slow decay (bad!)
- New goal: modify the design so that
  - faster decay of time-domain response is achieved
  - → use :
    - » more spectrum (*excess bandwidth*)
    - » less steep transition band
  - constrain zero ISI at sampling instants = *Nyquist criterion*

Nyquist criterion...

- Requirement for zero ISI:

$$h_{\mathrm{T}}(kT) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases}$$

- The same with delta functions:

$$h_{\mathrm{T}}(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT) = \delta(t)$$

Nyquist criterion...

- Convert into the frequency domain:

$$h_{\mathrm{T}}(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t - kT) = \delta(t)$$

$$\Leftrightarrow \quad H_{\mathrm{T}}(f) * \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(f - m/T) = 1$$

$$\Leftrightarrow \quad \frac{1}{T} \sum_{m=-\infty}^{\infty} H_{\mathrm{T}}(f - m/T) = 1$$

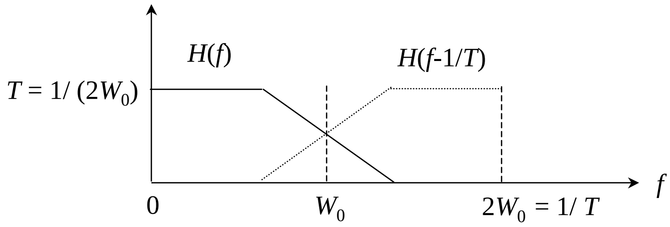
Nyquist criterion...

- General form of Nyquist criterion:

$$\frac{1}{T}\sum_{m=-\infty}^{\infty} H_T(f - m/T) = 1$$

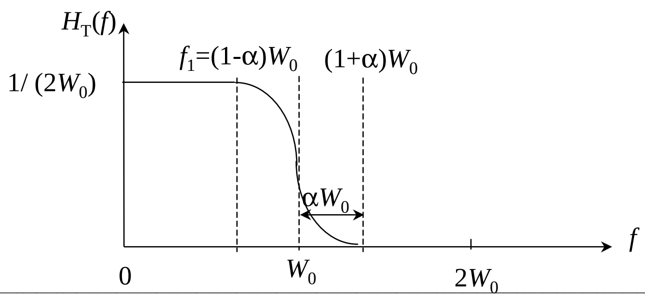
- For a bandlimited spectrum with $W < 2W_0 = 1/T$, we get

$$H_T(f) + H_T(f - 1/T) = T$$



Nyquist criterion...

- The Nyquist criterion does not define a unique spectrum
  $\rightarrow$ other constraints can be included (smooth transition!)
- Design parameter: *rolloff factor* α
- Determines used *excess bandwidth* αW_0



Nyquist criterion...

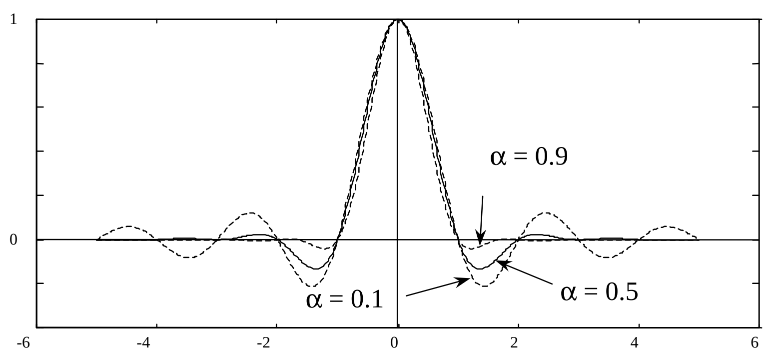
- Standard choice: Raised-Cosine (RC) spectrum:

$$H_T(f) = \begin{cases} 1/(2W_0), & |f| < f_1 \\ \frac{1}{4W_0}\left\{1+\cos\left[\frac{\pi(|f|-f_1)}{2W_0-f_1}\right]\right\}, & f_1 < |f| < 2W_0 - f_1 \\ 0, & |f| > 2W_0 - f_1 \end{cases}$$

with $f_1 = (1-\alpha)W_0$

Nyquist criterion...

- RC waveform via IDFT: $h_T(t) = \mathrm{sinc}(2W_0 t)\frac{\cos(2\pi\alpha W_0 t)}{1-(4\alpha W_0 t)^2}$



Nyquist criterion...

Observations on RC waveform:

- Pulse decays proportionally to $1/t^3$ and $1/\alpha^2$
- Increasing α:
  - uses more spectrum
  - improves pulse decay (α = 0: sinc pulse)
- Typical values in practice: α = 0.1…0.5

III. Matched filter

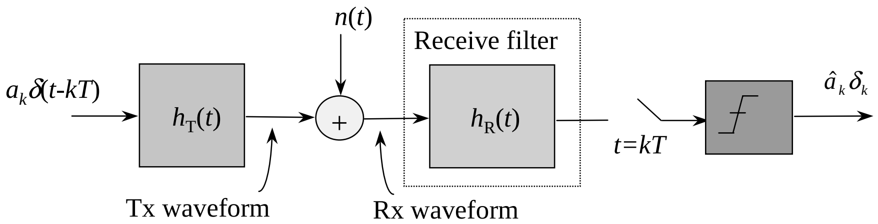
Matched filter

Consider *receiver* processing:

- If ISI is eliminated, *noise* is the main problem

Improve the system model:

- Add *receive filter* $h_R(t)$ to reduce noise



Matched filter...

- Known result: error probability in AWGN channel (no ISI):

$$P_e = Q(\sqrt{SNR})$$

→ Rx filter design criterion: maximize SNR !

- *Matched filter (MF):* maximizes SNR at the sampling instant for a given Tx pulse (usually in an AWGN channel)

Matched filter...

- Design optimal Rx filter $h_R(t)$ for a given Tx filter $h_T(t)$:
- Waveform after Rx filter (without noise):

$$g(t) = h_T(t) * h_R(t) = \int_{-\infty}^{\infty} H_T(f) H_R(f) e^{j2\pi ft} df$$

- Symbol energy at $t = 0$:

$$|g(0)|^2 = \left| \int_{-\infty}^{\infty} H_T(f) H_R(f) df \right|^2$$

- Schwarz inequality:

$$\left| \int_{-\infty}^{\infty} H_T(f) H_R(f) df \right|^2 \leq \int_{-\infty}^{\infty} |H_T(f)|^2 df \int_{-\infty}^{\infty} |H_R(f)|^2 df$$

Matched filter...

- Average noise power after receive filter:

$$E[n_R^2(t)] = \int_{-\infty}^{\infty} S_n(f) |H_R(f)|^2 df = \frac{N_0}{2} \int_{-\infty}^{\infty} |H_R(f)|^2 df$$

- Signal-to-noise ratio at sampling instant:

$$SNR = \frac{|g(0)|^2}{E[n_R^2(t)]} = \frac{\left| \int_{-\infty}^{\infty} H_T(f) H_R(f) df \right|^2}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H_R(f)|^2 df}$$

Matched filter...

- Use Schwarz inequality:

$$SNR \le \frac{\int_{-\infty}^{\infty} |H_T(f)|^2 df \int_{-\infty}^{\infty} |H_R(f)|^2 df}{\frac{N_0}{2} \int_{-\infty}^{\infty} |H_R(f)|^2 df} = \frac{2}{N_0} \int_{-\infty}^{\infty} |H_T(f)|^2 df = SNR_{MAX}$$

- The max SNR is obtained when

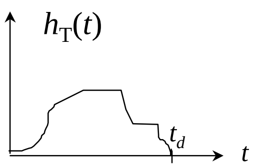
$$H_R(f) = k_0 H_T^*(f), \quad k_0 = \text{const.}$$

Matched filter...

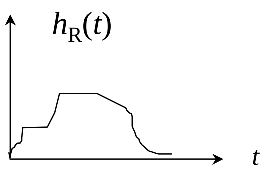
- Time-domain waveform of MF:

$$H_R(f) = k_0 H_T^*(f) \quad \Leftrightarrow \quad h_R(t) = k_0 h_T(-t)$$

- Causal implementation: $h_R(t) = k_0 h_T(t_d - t)$



Transmit filter



Matched receive filter

Matched filter...

How much does MF improve over direct sampling (DS)?

1) DS:

$$SNR_{\mathrm{DS}} = \frac{|h_{\mathrm{T}}(0)|^2}{N_0/2} = \frac{2}{N_0}, \quad h_{\mathrm{T}}(0) = 1$$

2) MF:

$$SNR_{\mathrm{MF}} = \frac{2}{N_0}\int_{-\infty}^{\infty}|H_{\mathrm{T}}(f)|^2\,df = \frac{2}{N_0}\int_{-\infty}^{\infty}|h_{\mathrm{T}}(t)|^2\,dt$$

$$\Rightarrow \frac{SNR_{\mathrm{MF}}}{SNR_{\mathrm{DS}}} = \int_{-\infty}^{\infty}|h_{\mathrm{T}}(t)|^2\,dt = E_{\mathrm{S}}$$

MF gain depends on transmitted symbol energy E_S!
Explain why!

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IV. Root-Nyquist filtering

Root-Nyquist filtering

Two criteria for Tx-Rx filter optimization

- Nyquist criterion (to guarantee zero ISI)
- Matched filter (to maximize SNR at the receiver)

NOTE! MF optimal only when no ISI!

How to combine the two?

Root-Nyquist filtering...

- Consider a matched pulse pair:

$$g(t) = h_T(t) * h_R(t) = h_T(t) * h_T(-t)$$

- Composite frequency response:

$$G(f) = H_T(f) H_T^*(f) = |H_T(f)|^2$$

$\rightarrow$ One possible design algorithm:

1) Design $G(f)$ as *real-valued & positive* Nyquist spectrum (e.g. raised-cosine)

2) Choose $H_T(f) = H_R(f) = \sqrt{G(f)}$

3) Get pulse waveform via IFT

Root-Nyquist filtering...

- Root-Raised-Cosine (RRC) spectrum:

$$H_T(f) = \begin{cases} 1/\sqrt{2W_0}, & |f| < f_1 \\ \frac{1}{\sqrt{4W_0}}\sqrt{1+\cos\left[\frac{\pi(|f|-f_1)}{2W_0-f_1}\right]}, & f_1 < |f| < 2W_0 - f_1 \\ 0, & |f| > 2W_0 - f_1 \end{cases}$$

with $f_1 = (1-\alpha)W_0$

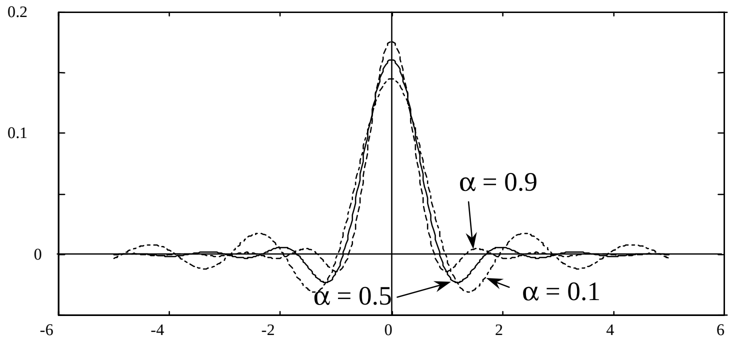
Root-Nyquist filtering...

- Root-Raised-Cosine (RRC) waveform via IFT (nice exercise for homework!):

$$h_T(t) = \frac{\sqrt{T}}{\pi t}\sin\left(2\pi(1-\alpha)W_0 t\right) + \frac{\sqrt{T}}{(2\pi t)^2 - \left(\frac{\pi}{4\alpha W_0}\right)^2}\Big[(4\pi t)\sin\left(2\pi(\alpha-1)W_0 t\right) - \left(\frac{\pi}{2\alpha W_0}\right)\cos\left(2\pi(\alpha+1)W_0 t\right)\Big]$$

Root-Nyquist filtering...

◆ RRC waveform plots for different α:



Root-Nyquist filtering...

Observations on RRC waveform:

◆ Similar to RC

◆ Except: *no* zeros at $t = kT$!
(Zeros come after convolution of Tx & Rx filter)

V. Tx and Rx filters in practice

Tx and Rx filters in practice

- The RRC Tx and Rx filters are (almost) optimal (= MIN P_e) for AWGN channels
- Linear channel destroys both zero ISI and MF property
- However, RRC is commonly used as Tx filter and as *preprocessor* at the receiver
- In addition, for ISI elimination we usually need
  - Linear Equalizer
  - DFE Equalizer or
  - Viterbi algorithm

Tx and Rx filters in practice...

Steps for digital Tx & Rx FIR filter design:

1) Continuous-time RRC from formula
2) Sample (usually 2-4 times symbol rate)
3) Truncate symmetrically (long enough!)
4) Quantize coefficients

Summary

Today we discussed:

Transmit and receive filters for bandlimited channels

I. Ideal sinc solution
II. Nyquist criterion
III. Matched filter
IV. Root-Nyquist filtering
V. Tx and Rx filters in practice

Next week: Optimal linear equalizers for linear channels I