Helsinki University of Technology
Signal Processing Laboratory

**S-38.411 Signal Processing in Telecommunications I**

Spring 2000

Lecture 2: Channel Capacity

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# Timetable

**L1** Introduction; models for channels and communication systems
**L2** **Channel capacity**
**L3** Transmit and receive filters for bandlimited AWGN channels
**L4** Optimal linear equalizers for linear channels 1
**L5** Optimal linear equalizers for linear channels 2
**L6** Adaptive equalizers 1
**L7** Adaptive equalizers 2
**L8** Nonlinear receivers 1: DFE equalizers
**L9** Nonlinear receivers 2: Viterbi algorithm
**L10 GL1**: DSP for Fixed Networks / *Matti Lehtimäki, Nokia Networks*
**L11 GL2**: DSP for Digital Subscriber Lines / *Janne Väänänen, Tellabs*
**L12 GL3**: DSP for CDMA Mobile Systems / *Kari Kalliojärvi, NRC*
**L13** Course review, questions, feedback
**E** 24.5. (Wed) 9-12 S4 **Exam**

# Contents of Lecture 2





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# I. Capacity of AWGN Channel

# Why capacity analysis?

- To design practical communication systems, one needs to understand the theoretical limits of transmission

Measures for transmission capacity (= max bit rate):

- Shannon capacity
  - based on information theory
  - maximum transmission rate which enables error-free transmission (in theory)
- Outage capacity
  - transmission capacity (or no. of users) that is available e.g. 95% of time (at prescribed error probability)
  - can be measured in practice
- Etc.



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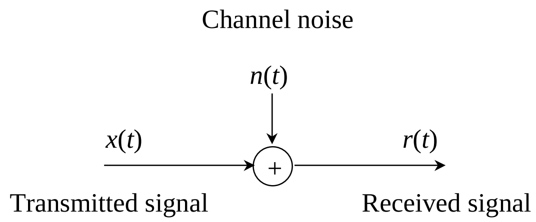
# Capacity of AWGN Channel

# Capacity of AWGN channel

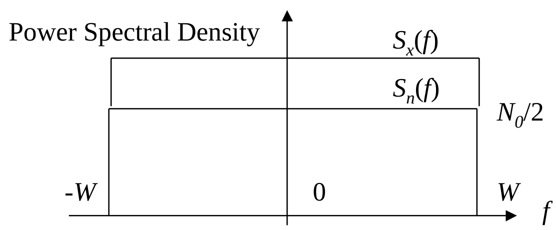



Bandlimitation:

- Every channel uses limited frequency band
- Limits both the signal and noise bandwidth




# Capacity of AWGN channel...




Distribution of signal and noise power in frequency:

- Assume flat noise power spectrum (AWGN)
- Assume signal bandwidth of *W* Hz

# Symbol Rate Limitation

- Sampling theorem: in order to accurately represent a signal with max frequency component of $W$ Hz, the sampling rate must be

$$f_s \geq 2W \qquad (2.1.)$$

- Conversely: in the bandwidth of $W$ Hz we can only represent a sampled signal of at most of the rate
  $R_{MAX} = 2W =$ *maximum symbol rate*



# Bits per Symbol Limitation

- How many bits can be loaded onto one symbol to be transmitted?
- Basic result from information theory (Shannon):

$$C_S = \frac{1}{2}\log_2\left(1 + \frac{P_x}{\sigma^2}\right) = \frac{1}{2}\log_2(1 + SNR) \quad \text{[bits/symbol]}$$

- Capacity per symbol in an AWGN channel depends on SNR only

# Total Transmission Capacity

- Maximum bit rate in AWGN channel:

$$C = R_{MAX} C_S = 2W \cdot \frac{1}{2} \log_2(1 + SNR) = W \log_2(1 + SNR)$$

- Also called the *Shannon Limit*
- Transmission capacity is maximized by
  - using as high symbol rate as possible
  - using a dense symbol constellation (as many bits/symbol as possible)



# Example: Telephone Channel Capacity

- Analog telephone lines use the frequency band of 300-3400 Hz and may have a typical SNR of 30 dB. Assuming ideal AWGN channel, what is the channel capacity?
- Solution:

$$SNR = 10^{30/10} = 1000$$

$$\begin{aligned} C &= W \log_2(1 + SNR) \\ &= 3.1\text{kHz} \times \log_2(1001) \\ &= 3.1\text{kHz} \times \ln(1001)/\ln(2) \\ &\approx 30.8\,\text{kbit/s} \end{aligned}$$

- How transmission rates of several Mbit/s with DSL techniques can be possible?

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# Capacity of Linear Channel with Colored Noise

## Linear Channel

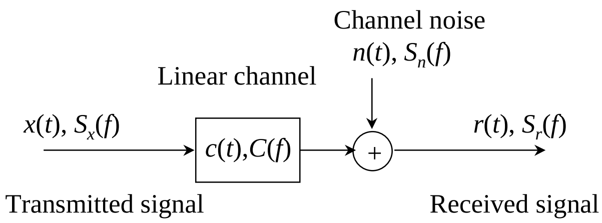
- AWGN channel model is not accurate for most channels
- Features to be added:

1) Linear distortion
   - different gain at different frequencies
   - pulse spreading in time domain (→ ISI, intersymbol interference)

2) Colored noise
   - non-flat noise power spectrum

→ Channel capacity is reduced

# Linear Channel...




- Autocorrelation function: $r_n(\tau) = E[n(t)n(t+\tau)]$

- Power spectrum (Power Spectral Density, PSD): $S_n(f) = \int_{-\infty}^{\infty} r_n(\tau) e^{-j2\pi f\tau} d\tau$



# Linear Channel...

- How does linear channel affect power spectrum?

$$r(t) = c(t) * x(t)$$

- By its squared magnitude:

$$S_r(f) = |C(f)|^2 S_x(f)$$

# Linear Channel...

- Consider the capacity of a small *slice* of linear channel (width $\Delta f$ at frequency $f_0$) :

$$C(f_0) = \Delta f \log_2\left(1 + \frac{S_x(f_0)|C(f_0)|^2 \Delta f}{S_n(f_0)\Delta f}\right)$$

- Obtain total capacity by *integration*:

$$C = \int_0^{\infty} \log_2\left(1 + \frac{S_x(f)|C(f)|^2}{S_n(f)}\right) df = \frac{1}{2}\int_{-\infty}^{\infty} \log_2\left(1 + \frac{S_x(f)|C(f)|^2}{S_n(f)}\right) df$$



# Linear Channel...

- Input PSD $S_x(f)$ needs to be known to evaluate capacity
  $\rightarrow$ $S_x(f)$ can be optimized to maximize capacity!
- Limited power constraint:

$$E\left[x^2(t)\right] = \int_{-\infty}^{\infty} S_x(f) df = P_x$$

- Constrained optimization via Lagrange multipliers
- Cost function:

$$g(S_x, \lambda) = C + \lambda\left\{P_x - \int_{-\infty}^{\infty} S_x(f) df\right\}$$

## Linear Channel...

- Our cost function:

$$g(S_x,\lambda)=\int_{-\infty}^{\infty}\left\{\tfrac{1}{2}\log_2\left(1+S_x(f)\left|C(f)\right|^2/S_n(f)\right)-\lambda S_x(f)\right\}df+\lambda P_x$$

- Optimization: solve for zeros of the derivatives

$$\begin{cases}\dfrac{\partial g}{\partial S_x}=\displaystyle\int_{-\infty}^{\infty}\left\{\frac{1}{2\ln 2}\cdot\frac{1}{1+S_x\left|C\right|^2/S_n}\cdot\frac{\left|C\right|^2}{S_n}-\lambda\right\}df=0\\ \dfrac{\partial g}{\partial\lambda}=P_x-\displaystyle\int_{-\infty}^{\infty}S_x(f)df=0\end{cases}$$



## Linear Channel...

- Solving for optimum $S_x(f)$ gives

$$S_{x,opt}(f)=\begin{cases}L-\dfrac{S_n(f)}{\left|C(f)\right|^2}, & f\in F\\ 0, & \text{elsewhere}\end{cases}$$

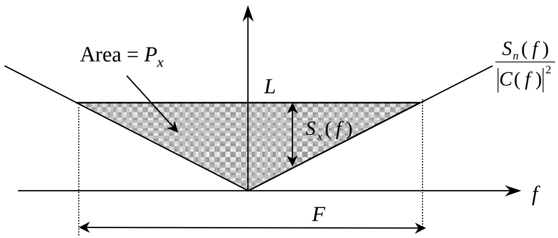
where

$$L=\frac{1}{\lambda 2\ln 2}$$

and $F$ is the frequency region where $S_x(f)$ is *positive*

# Linear Channel...

Water pouring theorem:




Where should Tx power be allocated to maximize capacity?

At frequencies where:

- Channel noise PSD is low
- Channel gain is large (low attenuation)



# Example 1: AWGN Channel

- Consider AWGN channel of bandwidth *W*:

$$S_n(f) = P_n / W, \quad 0 \le f \le W$$
$$|C(f)| = 1$$
$$S_x(f) = S_{x,opt}(f) = ?$$

- Solve for optimal Tx spectrum:

$$S_x(f) = L - S_n(f)/|C(f)|^2$$
$$= L - P_n / W$$

# Example 1: AWGN Channel...

- Constrained total transmit power:

$$P_x = \int_0^W S_x(f)df = (L - P_n / W)W$$
$$\Rightarrow L = (P_x + P_n)W$$
$$\Rightarrow S_{x,opt}(f) = P_x / W$$

- Hence, in the AWGN channel, the constant Tx power spectrum **IS** optimal!



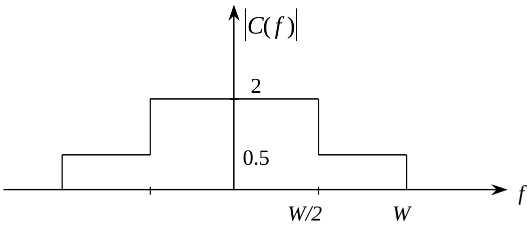
# Example 1: AWGN Channel

- Total capacity:

$$C = \int_0^W \log_2\left(1 + \frac{S_x(f)|C(f)|^2}{S_n(f)}\right)df$$
$$= W\log_2\left(1 + \frac{P_x / W \cdot 1}{P_n / W}\right)$$
$$= W\log_2(1 + SNR)$$

- Gives the expected result!

## Example 2: Two-band Channel




- Consider a two-band channel of bandwidth *W*:

$$S_n(f) = P_n / W \quad 0 \le |f| \le W/2$$

$$|C(f)| = \begin{cases} 2, & 0 \le |f| \le W/2 \\ 0.5, & W/2 \le |f| \le W \end{cases}$$

$$S_x(f) = S_{x,opt}(f) = ?$$

$$C = ?$$

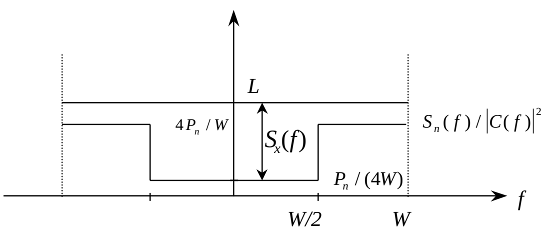


## Example 2: Two-band Channel...

- Optimal Tx power spectrum:

$$S_{x,opt}(f) = L - S_n(f)/|C(f)|^2 = \begin{cases} L - P_n/(4W), & 0 \le |f| \le W/2 \\ L - 4P_n/W, & W/2 \le |f| \le W \end{cases}$$

$$L = (P_x + 17P_n/8)/W$$



# Example 2: Two-band Channel...

- With some elaboration, the capacity can be solved as

$$
\begin{aligned}
C &= \int_0^W \log_2\left(1 + S_x(f)|C(f)|^2 / S_n(f)\right) df \\
&= \frac{W}{2}\left(\log_2\left(1 + \frac{P_x + \frac{15}{8}P_n \cdot 4}{P_n}\right) + \log_2\left(1 + \frac{P_x - \frac{15}{8}P_n \cdot \frac{1}{4}}{P_n}\right)\right) \\
&= \ldots = W \log_2\left(1 + SNR + \tfrac{9}{8}\right)
\end{aligned}
$$

- Compare with AWGN channel!
- Which one has higher capacity?



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# III. Capacity of Multiuser Channels

## Capacity of multiuser channels

In many applications, multiple users share the same channel ( = multiple-access channels)

- Mobile cellular communications systems
- Broadcast channels (TV, radio)
- Store-and-forward channels (satellite relays)



## Capacity of multiuser channels...

Different channel sharing strategies

- FDMA = Frequency Division Multiple Access
- TDMA = Time Division Multiple Access
- CDMA = Code Division Multiple Access

All are based on some (almost-)orthogonal division of users

# Capacity of multiuser channels...

- With ideal orthogonality, the total channel capacity can be divided to the users without losses
- In practice, losses are caused by
  - nonideal bandpass filters in FDMA
  - nonideal timing and time overlapping in TDMA
  - nonorthogonal codes in CDMA
  - intersymbol interference (ISI) and adjacent channel interference (ACI) caused by channel
- Special problems with asynchronous transmission

(need for traffic control!)



# Capacity of multiuser channels...

- For single-user channels with ISI, optimal *maximum-likelihood sequence detection* (MLSD) techniques can be derived which give (close to) optimum capacity
- For multiple-access channels with ACI, analogous *multiuser detection* (MUD) techniques can be derived which give (close to) optimum capacity for each user
- Computationally very intensive!

# Capacity of multiuser channels...

Suboptimal techniques based on *interference cancellation*:

- Detect interfering signals
- Reconstruct signals (including channel effect)
- Subtract interfering signal
- Detect desired signal



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# III. Other interference and effect on capacity

# Other interference and capacity

Communications systems often suffer from interference from other systems, like:

- 50 Hz power lines
- Radio amateurs (inductive coupling to telephone lines!)
- PCM systems to ADSL connections
- Household electric devices to cordless telephones
- etc. etc.



# Other interference and capacity...

- If interference is treated as noise, it reduces the capacity in the worst way
- By employing information about the the structure of interference, better results are obtained
- General multiuser detection (interference cancellation)

# Summary

Today we discussed:

Channel capacity

- I. Capacity of AWGN Channel
- II. Capacity of linear channel with coloured noise
- III. Capacity of multiuser channels
- IV. Other interference and effect on capacity

Next week:

- Transmit and receive filters for a bandlimited AWGN channel