D7/1 Consider a link in a circuit switched trunk network. Denote by n the number of parallel channels. Users generate new calls according to a Poisson process. The mean interarrival time between new calls is denoted by t, and the mean call holding time by h.

(a) What is the traffic model in question (with Kendall's notation)?

(b) Determine the time blocking, the call blocking, and the traffic carried for $n = 2$, $t = 4$ min, and $h = 3$ min.

D7/2 Consider a link in a circuit switched access network. Denote by n the number of parallel channels. There are k on-off type users generating new calls when idle, with $k > n$. The mean idle time is denoted by t, and the mean call holding time by h.

(a) What is the traffic model in question (with Kendall's notation)?

(b) Determine the time blocking, the call blocking, and the traffic carried for $n = 2$, $k = 4$, $t = 9$ min, and $h = 3$ min.

D7/3 Consider a pure loss system with two servers. Customers arrive in independent batches of size 1 or 2. Both sizes are equally probable. These batches arrive according to a Poisson process with intensity λ. The whole batch is lost whenever the system is full at the arrival time. But if exactly one of the servers is idle when a new batch of size 2 arrives, only one of the arriving customers is lost. The customers are served individually and independently with the service time following the $\text{Exp}(\mu)$ distribution. Let $X(t)$ denote the number of customers in the system at time t, which is a Markov process.

(a) Draw the state transition diagram of $X(t)$.

(b) Derive the equilibrium distribution of $X(t)$.

(c) Assume that $\lambda = \mu$. What is the utilization factor of the system, that is, the mean number of busy servers divided by the total number of servers?

(d) Assume again that $\lambda = \mu$. What is the "call" blocking probability, that is, the probability that an arriving customer is lost?

D7/1 (a) This is the M/G/n/n model, that is, the Erlang model with a general call holding time distribution (L7/15).

(b) Now the arrival rate is $\lambda == 1/t = 1/4$ calls/min, and the traffic intensity $a = \lambda h = h/t = 3/4 = 0.75$ erl. In the Erlang model, the call blocking B_{c} and the time blocking B_{t} are equal, and they can calculated from the Erlang formula (L7/20):

$$B_{\mathrm{c}} = B_{\mathrm{t}} = \frac{\frac{a^n}{n!}}{\sum_{j=0}^{n} \frac{a^j}{j!}} = \frac{\frac{1}{2}(\frac{3}{4})^2}{1 + \frac{3}{4} + \frac{1}{2}(\frac{3}{4})^2} = \frac{9}{32 + 24 + 9} = \frac{9}{65} = 0.14$$

Thus, the traffic carried is

$$a_{\mathrm{carried}} = a(1 - B_{\mathrm{c}}) = \frac{3}{4} \cdot (1 - \frac{9}{65}) = \frac{42}{65} = 0.65\,\mathrm{erl}$$

On the other hand, by Little's formula (L1/31), the traffic carried equals the mean number of customers in the system, $E[X]$. Since the equilibrium distribution of the Erlang model is the following truncated Poisson distribution (L7/18),

$$\pi_i = \frac{\frac{a^i}{i!}}{\sum_{j=0}^{n} \frac{a^j}{j!}}, \qquad i = 0, 1, 2,$$

the mean value $E[X]$ becomes

$$\begin{aligned} E[X] &= \sum_{i=0}^{n} i \cdot \pi_i = \frac{\frac{3}{4}}{1 + \frac{3}{4} + \frac{1}{2}(\frac{3}{4})^2} + 2 \cdot \frac{\frac{1}{2}(\frac{3}{4})^2}{1 + (\frac{3}{4}) + \frac{1}{2}(\frac{3}{4})^2} \\ &= \frac{24}{32 + 24 + 9} + 2 \cdot \frac{9}{32 + 24 + 9} = \frac{42}{65} = 0.65, \end{aligned}$$

as it should be.

D7/2 (a) This is the M/G/n/n/k model, that is, the Engset model with a general call holding time distribution (L7/32).

(b) When idle, a user becomes active with intensity $\nu = 1/t = 1/9$ times/min. Correspondingly, when active, a user becomes idle with intensity $\mu = 1/h = 1/3$ times/min. Thus, $\nu/\mu = 3/9 = 1/3 = 0.33$. The formula for the time blocking is given by (L7/36)

$$B_{\mathrm{t}} = \frac{\binom{k}{n}(\frac{\nu}{\mu})^n}{\sum_{j=0}^{n} \binom{k}{j}(\frac{\nu}{\mu})^j} = \frac{6(\frac{1}{3})^2}{1 + 4(\frac{1}{3}) + 6(\frac{1}{3})^2} = \frac{6}{9 + 12 + 6} = \frac{2}{9} = 0.22$$

The call blocking equals the time blocking in a modified system with one less customer, and can be calculated using the Engset formula (L7/40):

$$B_{\mathrm{c}} = \frac{\binom{k-1}{n}(\frac{\nu}{\mu})^n}{\sum_{j=0}^{n} \binom{k-1}{j}(\frac{\nu}{\mu})^j} = \frac{3(\frac{1}{3})^2}{1 + 3(\frac{1}{3}) + 3(\frac{1}{3})^2} = \frac{1}{3 + 3 + 1} = \frac{1}{7} = 0.14$$

By Little's formula (L1/31), the traffic carried equals the mean number of customers in the system, $E[X]$. Since the equilibrium distribution of the Engset model is the following truncated binomial distribution (L7/35),

$$\pi_i = \frac{\binom{k}{i}(\frac{\nu}{\mu})^i}{\sum_{j=0}^{n} \binom{k}{j}(\frac{\nu}{\mu})^j}, \qquad i = 0, 1, 2,$$

the mean value $E[X]$ becomes

$$\begin{aligned} E[X] &= \sum_{i=0}^{n} i \cdot \pi_i = \frac{4(\frac{1}{3})}{1+4(\frac{1}{3})+6(\frac{1}{3})^2} + 2 \cdot \frac{6(\frac{1}{3})^2}{1+4(\frac{1}{3})+6(\frac{1}{3})^2} \\ &= \frac{12}{9+12+6} + 2 \cdot \frac{6}{9+12+6} = \frac{8}{9} = 0.89 \end{aligned}$$

Thus, the traffic carried is $a_{\text{carried}} = E[X] = 0.89$ erl.
Note: Due to the finite population of the Engset model, there are two slightly different definitions for the offered traffic: the hypothetical offered traffic $a^{\text{h}}_{\text{offered}}$ and the realized offerd traffic $a^{\text{r}}_{\text{offered}}$. According to L7/35, the (hypothetical) offered traffic is in this case

$$a^{\text{h}}_{\text{offered}} = \frac{k\nu}{\nu+\mu} = \frac{k(\frac{\nu}{\mu})}{1+(\frac{\nu}{\mu})} = \frac{4(\frac{1}{3})}{1+(\frac{1}{3})} = 1.$$

So this is equal to the carried traffic in the corresponding lossless system (that is, the binomial model M/G/k/k/k). As it is reasonable to require, this characterization of the offered traffic is independent of the system parameters (such as the number of servers, n). However, the realized offered traffic is different from this due to the feedback mechanism of the model: the same customers return to the system (after an idle period). The (average) realized arrival rate is clearly

$$\lambda^{\text{r}} = \sum_{i=0}^{n} (k-i)\nu \cdot \pi_i = \cdots = \frac{28}{81},$$

so that the realized offered traffic becomes

$$a^{\text{r}}_{\text{offered}} = \lambda^{\text{h}}/\mu = \frac{28}{27} = 1.037.$$

This can be expressed in an equivalent form as follows:

$$a^{\text{r}}_{\text{offered}} = \frac{k\nu}{\nu(1-B_{\text{c}})+\mu} = \frac{k(\frac{\nu}{\mu})}{1+(\frac{\nu}{\mu})(1-B_{\text{c}})} = \frac{4(\frac{1}{3})}{1+(\frac{1}{3})(1-\frac{1}{7})} = \frac{28}{27} = 1.037.$$

Finally, the carried traffic satisfies

$$a_{\text{carried}} = a^{\text{r}}_{\text{offered}}(1-B_{\text{c}}) = \frac{28}{27}(1-\frac{1}{7}) = \frac{8}{9} = 0.89$$

D7/3 (a) Figure 1.

(b) We see from Figure 1 that the Markov process $X(t)$ is irreducible (L6/10). Since the state space is finite, the equilibrium distribution π exists, and it can be derived based on the global balance equations (GBE) and the normalization condition (N), cf. L6/11.

Let us start with the GBE's for states 0 and 2:

$$\pi_0\lambda = \pi_1\mu, \quad \pi_2\, 2\mu = \pi_0\,\frac{\lambda}{2} + \pi_1\lambda$$

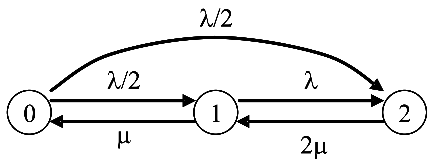

Figure 1: [D7/3] State transition diagram.

The other probabilities are now solved as a function of π_0:

$$\pi_1 = \pi_0 \frac{\lambda}{\mu}, \quad \pi_2 = \pi_0 \left(\frac{1}{4}(\frac{\lambda}{\mu}) + \frac{1}{2}(\frac{\lambda}{\mu})^2 \right)$$

The remaining probability π_0 is determined by (N):

$$\pi_0 + \pi_1 + \pi_2 = \pi_0 \left(1 + \frac{5}{4}(\frac{\lambda}{\mu}) + \frac{1}{2}(\frac{\lambda}{\mu})^2 \right) = 1.$$

Thus, the equilibrium distribution is

$$\pi_0 = \frac{1}{1 + \frac{5}{4}(\frac{\lambda}{\mu}) + \frac{1}{2}(\frac{\lambda}{\mu})^2}, \quad \pi_1 = \frac{\frac{\lambda}{\mu}}{1 + \frac{5}{4}(\frac{\lambda}{\mu}) + \frac{1}{2}(\frac{\lambda}{\mu})^2}, \quad \pi_2 = \frac{\frac{1}{4}(\frac{\lambda}{\mu}) + \frac{1}{2}(\frac{\lambda}{\mu})^2}{1 + \frac{5}{4}(\frac{\lambda}{\mu}) + \frac{1}{2}(\frac{\lambda}{\mu})^2}$$

If $\lambda = \mu$ (as will be assumed in (c) and (d)), the equlibrium distribution is

$$\pi_0 = \frac{4}{11} = 0.36, \quad \pi_1 = \frac{4}{11} = 0.36, \quad \pi_2 = \frac{3}{11} = 0.27$$

(c) The mean number of busy servers is

$$E[X_\mathrm{s}] = \sum_{i=0}^{n} i \cdot \pi_i = \frac{4}{11} + 2 \cdot \frac{3}{11} = \frac{10}{11} = 0.91$$

Thus, the utilization factor becomes

$$E[U] = \frac{E[X_\mathrm{s}]}{n} = \frac{(\frac{10}{11})}{2} = \frac{5}{11} = 0.45$$

(d) To calculate the call blocking probability, we need to know the mean number of customers in a batch, $E[A]$, and the mean number of lost customers in a batch, $E[L]$. The former one is clearly

$$E[A] = 1 \cdot P\{A = 1\} + 2 \cdot P\{A = 2\} = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2} = 1.50$$

On the other hand, due to the PASTA property of the Poisson process (L5/28), an arriving batch sees the system in equilibrium. Thus,

$$\begin{aligned} E[L] &= \pi_1 (1 \cdot P\{A = 2\}) + \pi_2 (1 \cdot P\{A = 1\} + 2 \cdot P\{A = 2\}) \\ &= \pi_1 P\{A = 2\} + \pi_2 E[A] = \frac{4}{11} \cdot \frac{1}{2} + \frac{3}{11} \cdot \frac{3}{2} = \frac{13}{22} = 0.59 \end{aligned}$$

The call blocking probability B_{C} is their ratio:

$$B_{\mathrm{C}} = \frac{E[L]}{E[A]} = \frac{\frac{13}{22}}{\frac{3}{2}} = \frac{13}{33} = 0.39$$

Note: The traffic intensity is now

$$a = \frac{\lambda E[A]}{\mu} = E[A] = \frac{3}{2} = 1.50$$

If the customers arrived individually (and not in batches) according to a Poisson process, the blocking probability would be, by the Erlang formula (L7/20),

$$\mathrm{Erl}(n, a) = \frac{\frac{a^n}{n!}}{\sum_{j=0}^{n} \frac{a^j}{j!}} = \frac{\frac{1}{2}(\frac{3}{2})^2}{1 + \frac{3}{2} + \frac{1}{2}(\frac{3}{2})^2} = \frac{9}{8 + 12 + 9} = \frac{9}{29} = 0.31,$$

which is less than for the original system. So a burstier arrival process results in a higher blocking probability.